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Computational Modeling of High-Order Differential Equations Using Radial Basis Functions and Numerical Optimization

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ABSTRACT

This paper presents a meshless computational framework for approximating solutions of ordinary differential equations, with a formulation suitable for extension to higher-order problems. The framework is based on radial basis functions and optimization techniques. The approach uses radial basis functions (RBFs) as a meshless approximation scheme to obtain accurate numerical solutions to initial and boundary value problems in a grid-free manner. In addition to shortening convergence behavior and enhancing computational performance, the L-BFGS scheme is introduced into the framework to derive stable numerical approximations to the governing differential equations. A structured meshless formulation for approximating ordinary differential equations is provided by the present framework, which lays the basis for its extension to higher-order problems. According to numerical experiments, the method accurately approximates solutions in the computational domain and provides stable extrapolations at the boundary. The numerical results give the first validation of the framework on smooth benchmark problems with known exact solutions. Furthermore, the formulation reduces dependence on predefined discretization strategies, supports flexible placement of collocation points, and enables analytical derivative evaluation, making it suitable for smooth benchmark problems and future extensions to more complex nonlinear and multidimensional differential systems in engineering.

الخلاصة

تقدّم هذه الورقة البحثية إطاراً حسابياً بلا شبكة لتقريب حلول المعادلات التفاضلية العادية، بصياغة قابلة للامتداد إلى المسائل عالية الرتبة. ويستند هذا الإطار إلى دوال الأساس الشعاعي وتقنيات التحسين العددي. توظّف المنهجية المقترحة دوال الأساس الشعاعي (RBFs) بوصفها أسلوب تقريب عددي بلا شبكة، بهدف الحصول على حلول عددية دقيقة لمسائل القيم الابتدائية والحدية دون الحاجة إلى إنشاء شبكة حسابية محددة مسبقاً. ولتحسين سلوك التقارب وتعزيز الأداء الحسابي، تُدمج خوارزمية L-BFGS ضمن الإطار المقترح لاشتقاق تقريبات عددية مستقرة للمعادلات التفاضلية الحاكمة. ويوفّر الإطار المقترح صياغة منظمة بلا شبكة لتقريب حلول المعادلات التفاضلية العادية، كما يشكّل أساساً قابلاً للامتداد إلى المسائل عالية الرتبة. وتبيّن التجارب العددية أن الطريقة تحقق تقريبات دقيقة داخل المجال الحسابي، وتُظهر سلوكاً مستقراً في مناطق الاستقراء قرب الحدود. وتوفّر النتائج العددية تحققاً أولياً من الإطار المقترح على مسائل معيارية ملساء ذات حلول دقيقة معروفة. علاوة على ذلك، تقلل الصياغة المقترحة من الاعتماد على استراتيجيات التجزئة المحددة مسبقاً، وتدعم التوزيع المرن لنقاط التجميع، وتتيح حساب المشتقات تحليلياً، مما يجعلها مناسبة للمسائل المعيارية الملساء وللتوسعات المستقبلية نحو أنظمة تفاضلية لا خطية ومتعددة الأبعاد أكثر تعقيداً في التطبيقات الهندسية.

1. Introduction

Numerical techniques including predictor–corrector schemes, Runge–Kutta algorithms, finite difference methods, finite element methods, finite volume techniques, boundary element methods, spectral methods, and collocation methods are typically used in applied mathematics and engineering to solve many real-world problems. Many physical or real-world phenomena can be modeled through differential equations along with initial or boundary conditions. Conventional numerical methods such as finite difference and finite element methods find approximate solutions by discretizing the computational domain into a chosen mesh and formulating numerical operators for derivatives (Alkhatib, 2025, pp. 2999–3013). Despite their usefulness, these methods become less flexible and more computationally expensive for high-order differential equations due to their reliance on mesh generation and discretization methods.

In order to overcome these limitations, meshless approximation techniques have gained much attention. Radial Basis Functions (RBFs), as one of the numerous techniques, allow for an adaptable computing structure that requires no previously known discretization grids for solution approximation. These limitations motivate the development of efficient meshless numerical frameworks capable of producing accurate and stable approximations while reducing computational complexity (Wendland, 2017, pp. 265–299).

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Several previous studies have contributed to the development of meshless numerical methods and optimization-based approaches for solving differential equations. Micchelli (Micchelli, 1986, pp. 11–22) established fundamental interpolation results for Radial Basis Functions, which later became a cornerstone of meshfree approximation techniques. Buhmann (Buhmann, 2003, pp. 48–98) developed the theoretical and computational properties of RBFs, while Fasshauer (Fasshauer, 2007, pp. 345–352) extended their practical use in meshless numerical frameworks. In addition, Lagaris et al. (Lagaris et al., 1998, pp. 987–1000) showed that trial-function-based and optimization-based formulations can provide accurate numerical solutions for ordinary and partial differential equations, while Meade and Fernandez (Meade & Fernandez, 1994, pp. 1–25) demonstrated a related feedforward-neural-network approach for solving linear ordinary differential equations. Furthermore, Nocedal and Wright (Nocedal & Wright, 2006, pp. 176–180) provided the theoretical basis for modern gradient-based optimization methods, including the L-BFGS algorithm.

Although classical discretization-based methods and the widespread use of the meshless approximation for differential equations have been used, these methods have particular computational limitations. For instance, limitations could be related to mesh dependence, approximating derivatives or sensitivity to values, optimization efficiencies, and the like. For these reasons, a new meshless framework is proposed in this paper, which combines RBF approximation with residual-based optimization.

This study's main scientific contribution is to develop an integrated residual-based meshless computational framework for obtaining the approximate solution of ordinary differential equations. The intended contribution, therefore, is not a new RBF kernel or a new optimization algorithm, but rather the systematic coupling of Gaussian RBF trial approximation with L-BFGS parameter optimization within a single numerical procedure. The approach consists of constructing a trial solution satisfying the prescribed initial or boundary conditions as well as computing the analytical derivative required by the differential operator, formulating a regularized residual-based objective function, and optimizing the RBF parameters using the L-BFGS algorithm. The study also confirms this contribution numerically by checking the effects of the number of RBF units, collocation points, and shape parameter on the error of the approximation. The newness of the undertaking is in the organized computational formulation and its initial numeric assessment as a base for extension to higher-order differential equations.

With this input, the mathematical formulation of the proposed RBF–L-BFGS framework is presented, the residual-based optimization procedure is discussed, and its numerical behavior is evaluated through benchmark problems possessing known exact solutions.

2. Radial Basis Functions and Collocation Framework

Radial Basis Functions help to approximate the solution without a grid. They achieve this through a weighted sum of radially symmetric kernel functions. In addition, these functions are centered at selected collocation points. (Fasshauer, 2007, pp. 345–352). The value of each kernel depends solely on the distance between the evaluation point and the center. This property allows the solution to be approximated without constructing a fixed computational mesh. In the proposed framework, Gaussian RBFs are used because of their smoothness and the availability of analytical derivatives. The shape parameter controls the flatness of the basis functions and has a direct influence on both approximation accuracy and numerical stability.

The foundation of the RBF-based approximation is constructed through the expansion:

$$N(x,P) = \sum_{i=1}^{H} \alpha_i \phi(\| x - c_i \|) \qquad \cdots (1)$$

Where $\phi(.)$ denotes a radial basis kernel, such as the Gaussian kernel:

$$\phi(r) = e^{-(\epsilon r)^2} \qquad \cdots (2)$$

The multiquadric kernel:

$$\phi(r) = \sqrt{1 + (\epsilon r)^2} \qquad \cdots (3)$$ or the thin-plate spline kernel

$$\phi(r) = r^2 lnr \qquad \cdots (4)$$

The unknown weights are represented by the coefficients. α_i represent, and the collocation centers by c_i. The radial basis function's flatness is controlled by the shape parameter ϵ, which affects the accuracy of approximation.

The parameters $P = \{\alpha_i, c_i, \epsilon\}$ in the set P are optimized such that the governing differential equation, along with its initial or boundary conditions, is satisfied.

Micchelli's theorem proves that the interpolation systems related to distinct interpolation nodes associated with conditionally positive definite radial functions are solvable. This ensures the existence and uniqueness of the RBF interpolation solution with the required side conditions (Micchelli, 1986, pp. 11–22). The overall accuracy of approximations varies for different problems. It depends primarily on the number of collocation centers H used, which, in turn, depends on the selected radial kernel. The shape parameter ε also has a significant impact on the accuracy of approximation.

The structure of the trial solution $yT(x,P)$in the hybrid framework is constructed so that the prescribed initial or boundary conditions are satisfied by design. This can be guaranteed by a combination of two terms: an auxiliary analytic function satisfying the given conditions and an RBF correction scaled by a weight that vanishes at the boundary. Therefore, the optimization process primarily concerns itself with minimizing the residual of the differential equation within the domain, while the boundary or initial constraints are preserved automatically.

For a typical boundary value problem, the trial solution is expressed as:

$$yT(x,P) = a(x) + \beta(x) \sum_{i=1}^{H} \alpha_i \phi(\| x - ci \|) \qquad \cdots (5)$$

The function $a(x)$ is crafted using analytical means to accurately fulfill the boundary conditions, whereas the $\beta(x)$ is a smooth cutoff function that vanishes at the boundary of the computational domain. By using this weighting function, the RBF expansion will meet the provided boundary conditions. In the case of a fourth-order boundary value problem, the weighting function can be selected as:

$$\beta(x) = (x-a)^2(x-b)^2 \qquad \cdots (6)$$

To ensure that the required boundary conditions are satisfied (Cao et al., 2023, pp. 1–11). In the proposed numerical framework, the coefficients $a(x)$, collocation centers c_i and shape parameter ε are optimized to yield an accurate solution of the governing differential equation.

The optimization phase is formulated as a minimization of the loss function $E(P)$. This loss function is the sum of square residuals of the governing differential equation, obtained by evaluating the

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governing differential equation at the collocation points $\{xi\}_{i=1}^{m}$). The objective function is presented in:

$$E(P)=\sum_{i=1}^{m}\left[L\left(x_i, yT(x_i,P),\frac{dyT}{dx},\dots,\frac{d^n yT}{dx_n}\right)\right]^2+\lambda\sum_{i=1}^{H}\alpha_i^2, \quad (7)$$

Where (L) denotes the differential operator that drives the governing equation, and the regularization term $\lambda\sum_{i=1}^{H}\alpha_i^2$ is included as a measure to alleviate ill-conditioning effects often found in RBF interpolation systems and to boost numerical stability during the optimization process. The value of λ is the regularization coefficient, and it controls the trade-off between the accuracy of the approximation and a smooth solution.

The proposed framework has, as a notable advantage, the availability of analytic derivatives of the radial basis kernels, enabling easy and accurate computation of high-order differential residuals. In fact, it is possible to evaluate the derivatives of the Gaussian RBF analytically. The k-th derivative is given by:

$$\frac{d^k\phi}{dx^k}=(-1)^k(2\epsilon^2)^k e^{(-\epsilon 2(x-ci)^2} H_k(\epsilon(x-c_i)), \quad (8)$$

Where H_k signifies the k-th Hermite polynomial (Soleymani et al., 2021, pp. 1–22). The presence of closed-form expressions for derivatives enhances the computational efficiency of the proposed optimization framework and aids in dealing with high-order differential equations.

The L-BFGS Algorithm is a limited-memory quasi-Newton algorithm, and in this paper, we carry out iterative minimization of the objective function through the gradient norm updating ($\nabla_P E$). Because L-BFGS accurately approximates second-order curvature information, it converges much faster than a derivative-free algorithm such as Nelder–Mead on smooth, high-dimensional optimization problems. (Rafati & Marica, 2020, pp. 9–38).

The use of trial-function-based and optimization-based formulations has been previously shown to provide accurate numerical solutions for ordinary and partial differential equations (Lagaris et al., 1998, pp. 987–1000). Accordingly, the present framework should be understood as a structured integration of RBF-based trial approximation, analytical residual construction, and L-BFGS-based parameter optimization, rather than as a replacement for the individual theories of RBF interpolation or quasi-Newton optimization.

3. Radial Basis Functions and Collocation Framework

We study a general boundary value problem of n-th order nonlinear defined on a closed interval ($x \in [a,b]$), given by:

$$D\left(x, y(x), y'(x), y''(x),\dots,y^{(n)}(x)\right)=0,\ \ x\in[a,b] \quad (9)$$

subject to the associated boundary and/or initial conditions:

$$C(y(x), y'(x), y''(x),\dots,y^{(n-1)}(x))\big|_{x=a,b}=0 \quad (10)$$

Where $D(.)$ denotes a general nonlinear differential operator of order n, and $C(.)$ represents the operator that enforces the prescribed boundary and/or initial conditions. The unknown function $y(x)$ is assumed to be sufficiently smooth on $[a,b]$ to ensure the existence of all required derivatives up to order n.

3.1 Trial Solution Based on Radial Basis Functions (RBFs):

The approximate solution $yT(x,P)$is formulated using a meshless radial basis function representation as follows:

$$yT(x,P)=a(x)+\beta(x)\sum_{i=1}^{H}\alpha i\,\phi(\|x-ci\|), \quad (11)$$

where $a(x)$ is an auxiliary analytic function constructed to satisfy the prescribed initial or boundary conditions exactly. The function $\beta(x)$ is a smooth weighting function designed to vanish at the boundary points, thereby preserving the imposed constraints. For instance, in fourth-order boundary value problems, a typical choice is:

$$\beta(x)=(x-a)^2(x-b)^2 \quad (12)$$

The function ϕ denotes the selected radial basis kernel, while αi represent the unknown expansion coefficients and ci correspond to the RBF centers (or collocation points). The parameter set is defined by.

This formulation combines the universal approximation capability of RBFs with the exact enforcement of boundary conditions through $a(x)$and $\beta(x)$ yielding a flexible and fully mesh-independent numerical representation suitable for solving high-order differential equations.

3.2 Optimization Objective Function:

The solution procedure is transformed into an unconstrained optimization problem in which the parameter vector P is determined by minimizing the residual of the governing differential equation evaluated at a set of collocation points. $\{x_i\}_{i=1}^{m}$The corresponding objective function is defined as

$$\min_P\sum_{i=1}^{m}||D[x_i, y_T(x_i,P),\frac{dy_T}{dx},\dots,\frac{d^n y_T}{dx^n}]||^2+\lambda\sum_{i=1}^{H}{\alpha_i}^2, \quad (13)$$

Subject to the boundary or initial constraints:

$$C\left[x, y_T(x,P),\frac{dy_T}{dx},\dots,\frac{d^n y_T}{dx^n}\right]=0\ ;\ x=a,b \quad (14)$$

Here, $D[.]$ denotes the governing differential operator, while $C[\cdot]$ represents the associated boundary or initial condition operator. The second term in the objective function, $\lambda\sum{\alpha_i}^2$ is a Tikhonov-type regularization term introduced to improve numerical stability and mitigate the ill-conditioning commonly associated with RBF interpolation matrices. The regularization parameter $\lambda>0$ controls the trade-off between accuracy and smoothness.

Through this formulation, the original differential equation problem is converted into a stable nonlinear optimization framework that can be efficiently solved using gradient-based optimization techniques such as the L-BFGS algorithm.

3.3 Analytical Computation of Derivatives:

A major advantage of the proposed radial basis function framework is the availability of closed-form analytical derivatives, which enables accurate and efficient evaluation of high-order differential operators without relying on numerical differentiation schemes (Li et al., 2015, pp. 2–9). Since the trial solution y_T (x,P) is expressed in terms of smooth radial basis kernels, its derivatives of arbitrary order can be computed analytically in a straightforward manner.

For the Gaussian radial basis function defined by

$$\phi(r)=e^{(-\varepsilon r)^2}: \quad \cdots(15)$$

The k-th order derivative with respect to x is given by:

$$\frac{d_k\phi}{dx^k}=(-1)k\,(2\varepsilon^2)k\,e^{-\varepsilon^2(x-c_i)^2}H_k(\varepsilon(x-c_i)), \quad (16)$$

Where $H_k(.)$ denotes the k-th Hermite polynomial and ε is the shape parameter controlling the smoothness of the kernel.

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The availability of analytical derivatives greatly enhances numerical accuracy and computational efficiency in solving high-order differential equations that make recurrent use of such higher-order derivatives. Additionally, this property avoids truncation errors encountered with finite-difference approximations and stabilizes the optimization process.

3.4 Optimization Using the L-BFGS Algorithm:

The L-BFGS algorithm is used to optimize the unknown RBF parameters by minimizing the residual-based objective function. The L-BFGS optimization technique leverages gradient information, enabling it to estimate curvature behaviour efficiently in limited memory, unlike derivative-free optimization methods (Nocedal & Wright, 2006, pp. 176–180). This is well suited for problems containing several RBF parameters (weights, centers, shape parameters). The use of analytical gradients further improves convergence and avoids the additional errors that may arise from numerical differentiation.

The objective function gradients guide the optimization process efficiently; E(P) with respect to the model parameters is computed analytically. Let

$$Residual(x_j) = D\left[x_j, y_T(x_j, P), \frac{dy_T}{dx}, \dots, d^n\frac{y_T}{dx^n}\right] \qquad (17)$$

Denote the residual of the governing differential equation evaluated at the collocation point x_j. The gradient components are then expressed as follows.

The derivative of the objective function with respect to the expansion coefficients α_i is given by:

$$\frac{\partial E}{\partial \alpha_i} = 2\sum_{j=1}^{m} Residual(x_j)\cdot \beta(x_j)\phi\left(\left\|x_j - c_i\right\|\right) + 2\lambda\alpha_i, \ (18)$$

Where the second term arises from the regularization component included to improve numerical stability.

Similarly, the gradient with respect to the RBF centers c_i is computed as

$$\frac{\partial E}{\partial c_i} = 2\sum_{j=1}^{m} Residual(x_j)\cdot \alpha_i\frac{\beta(x_j)\partial\phi}{\partial c_i}, \qquad \cdots (19)$$

While the derivative with respect to the shape parameter ε is defined by

$$\frac{\partial E}{\partial \varepsilon} = 2\sum_{j=1}^{m} Residual(x_j)\cdot \sum_{i=1}^{H}\alpha_i\frac{\beta(x_j)\partial\phi}{\partial\varepsilon} \qquad \cdots (20)$$

The availability of explicit analytical gradients significantly enhances the efficiency and robustness of the optimization procedure by eliminating the need for numerical differentiation (Lichtenecker et al., 2024, pp.153–164). Consequently, the L-BFGS algorithm can exploit accurate first-order information to achieve rapid and stable convergence, particularly for high-dimensional parameter spaces associated with meshless radial basis function approximations.

4. General Framework for Solving High-Order Differential Equations Using Radial Basis Functions

The proposed RBF–L-BFGS framework can be formulated for ordinary differential equations of different orders by constructing trial solutions that satisfy the prescribed initial or boundary conditions exactly. In this approach, the unknown component of the solution is approximated using Gaussian radial basis functions, while the residual of the governing differential equation is evaluated at selected collocation points.

The required derivatives of the RBF approximation are computed analytically for higher-order problems, allowing evaluation of the differential operator without numerical differentiation. We optimize the RBF parameters by minimizing our RBF-based objective function using the L-BFGS algorithm. The developed formulation can also be extended to higher-order initial and boundary value problems, although all the numerical validation is presented for a benchmark problem having an exact solution in this study.

4.1 Optimization Using the L-BFGS Algorithm:

To demonstrate the applicability of the suggested framework to high-order problems, a fourth-order problem is considered:

$$\frac{d^4y(x)}{dx^4} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right) \qquad \cdots \ (21)$$

subject to the boundary conditions:

$$y(a) = A; y(b) = B; y'(a) = A_0; y'(b) = B_0; x \in [a,b] \quad (22)$$

The trial solution is constructed in the form:

$$y_T(x,P) = Z(x) + M(x)N(x,P) \qquad (23)$$

Z(x) is chosen to satisfy the given boundary conditions, with M(x) being a smooth weighting function vanishing at the boundary points and N (x, P) being the RBF approximation. A suitable weighting function is given by:

$$M(x) = (x-a)^2\,(x-b)^2 \qquad \cdots (24)$$

The RBF correction term is constructed so that it satisfies the boundary conditions imposed. The L-BFGS algorithm estimates the undefined RBF parameters by minimizing the residual of the governing differential equation.

5. Numerical Results and Validation

The numerical experiment is intended to validate the basic accuracy and convergence behavior of the proposed RBF–L-BFGS framework. A benchmark problem with a known exact solution is chosen in such a way that the approximation error can be measured directly. The results indicate that the proposed method is highly accurate for the preferred smooth test problem, and that the accuracy is increased with the augmenting number of RBF basis functions and collocation points. Nonetheless, given that the benchmark problem at hand in the numerical section is first order, this analysis must be taken as an initial validation of the framework rather than a complete proof of performance for high-order differential equations.

5.1 Benchmark Problem and Mathematical Formulation:

Considering the first-order differential equation whose exact solution is known:

$$y(x) = x^4 - x^3 + 2x \; for \; x \in [0,1] \qquad (25)$$

The differentiation helps to obtain the governing differential equation as:

$$\frac{dy(x)}{dx} = 4x^3 - 3x^2 + 2 \; ; y(\mathbf{0}) = \mathbf{0} \qquad (26)$$

In order to fulfill the initial condition exactly, the trial solution is taken as:

$$y_T(x,P) = x \cdot N(x,P)$$

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$N(x, P)$ refers to the approximation of the RBF network defined by:

$$N(x, P) = x \sum_{i=1}^{H} vi \cdot \exp\left(-\varepsilon^2 (x - c_i)^2\right) \quad (27)$$

The adjustable parameters, namely the RBF centres, shape parameters, and expansion coefficients, are designated as $P = \{w, v, b\}$.

A minimization problem is formulated from the differential equation using a residual function:

$$R(x_i, P) = \frac{dyT(x_i, P)}{dx} - 4x_i^3 + 3x_i^2 - 2 \quad \cdots (28)$$

Thus, the objective functions are written as:

$$E(P) = \sum_{i=1}^{m} [R(x_i, P)]^2 \quad \cdots (29)$$

The derivative of the trial solution is obtained analytically as:

$$\frac{dyT}{dx} = N(x, P) + x.\frac{dN(x, P)}{dx} \quad \cdots (30)$$

where

$$\frac{dN(x, P)}{dx} = \sum v_i[-2\varepsilon^2(x - c_i)^2] \exp(-\varepsilon^2(x - c_i)^2) \quad (31)$$

The L-BFGS algorithm optimizes the parameter vector P by minimizing E(P). The closed form of derivatives of the Gaussian RBF ensures stable convergence and efficient function evaluation without numerical differentiation.

5.2 Numerical Results and Discussion:

The following first-order initial value problem was assumed to analyze the performance of the radial basis function (RBF) framework:

$$\frac{dy(x)}{dx}(x) = 4x^3 - 3x^2 + 2, \qquad x \in [0,1]\, ; y(0) = 0 \quad (32)$$

whose exact solution is provided by:

$$y(x) = x^4 - x^3 + 2x \quad \cdots (33)$$

An RBF expansion with Gaussian kernels of the type was used to approximate the numerical solution:

$$\phi(r) = e^{-(\varepsilon r)^2} \quad \cdots \quad (34)$$

Where ε is the shape parameter that controls the smoothness of the basis functions. A trial solution is known as:

$$yT(x) = \sum_{i=1}^{N} w_i\, \phi(x - c_i) \quad \cdots (35)$$

Where w_i are unknown weights and the centers c_i are uniformly distributed over the interval [0,1]. The following constraint is applied to enforce the original y (0) = 0 condition:

$$\sum_{i=1}^{N} w_i\, \phi(-c_i) = 0 \quad \cdots (36)$$

5.3 Collocation Formulation:

The governing differential equation is upheld at a collection of collocation points. $x_j \in [0,1]$, leading to:

$$\sum_{i=1}^{N} w_i \frac{d}{dx} \phi(x_j - c_i) = 4x_j^3 - 3x_j^2 + 2 \quad \cdots (37)$$

The Gaussian kernel's derivative is given analytically by:

$$\frac{d}{dx} \phi(x - c_i) = -2\varepsilon^2 (x - c_i)\phi(x - c_i) \quad \cdots (38)$$

Combining the boundary condition with the collocation equations transforms the problem into a linear algebraic system of the form:

$$AW = F \quad \cdots (39)$$

$$A = \begin{bmatrix} \phi_1(0) & \phi'_2(0) \ldots & \phi'_N(0) \\ \phi'_1(x_1) & \phi'_2(x_1) \ldots & \phi'_N(x_1) \\ . & . & . \\ . & . & . \\ \phi'_1(x_M) & \phi'_2(x_M) & \phi'_N(x_M) \end{bmatrix} \quad \cdots (40)$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_N \end{bmatrix} \quad \cdots (41)$$

$$F = \begin{bmatrix} 0 \\ 4x_1^3 - 3x_1^2 + 2 \\ . \\ . \\ 4x_M^3 - 3x_M^2 + 2 \end{bmatrix} \quad \cdots (42)$$

In this context, a stands for the evaluations of the RBFs and their derivatives at the boundary and collocation points, W denotes the unknown weight vector, while F refers to the forcing terms in the differential equation.

5.4 Collocation Formulation:

The approximation error is defined as the difference between the exact and the RBF-based approximate solution:

$$E(x) = y_{exact}(x) - y_T(x) \quad \cdots (43)$$

or explicitly:

$$E(x) = x^4 - x^3 + 2x - \sum_{i=1}^{N} w_i\, e^{-(\varepsilon(x - c_i))^2} \quad \cdots (44)$$

This error function allows us to assess the quality of the approximation of the proposed RBF formulation (Press et al., 2007, pp. 8–11). The L-BFGS optimization algorithm was employed to minimize the resultant objective function. The implementation of the model used H=5 basis units and m=6 equally spaced collocation points in the range [0,1].

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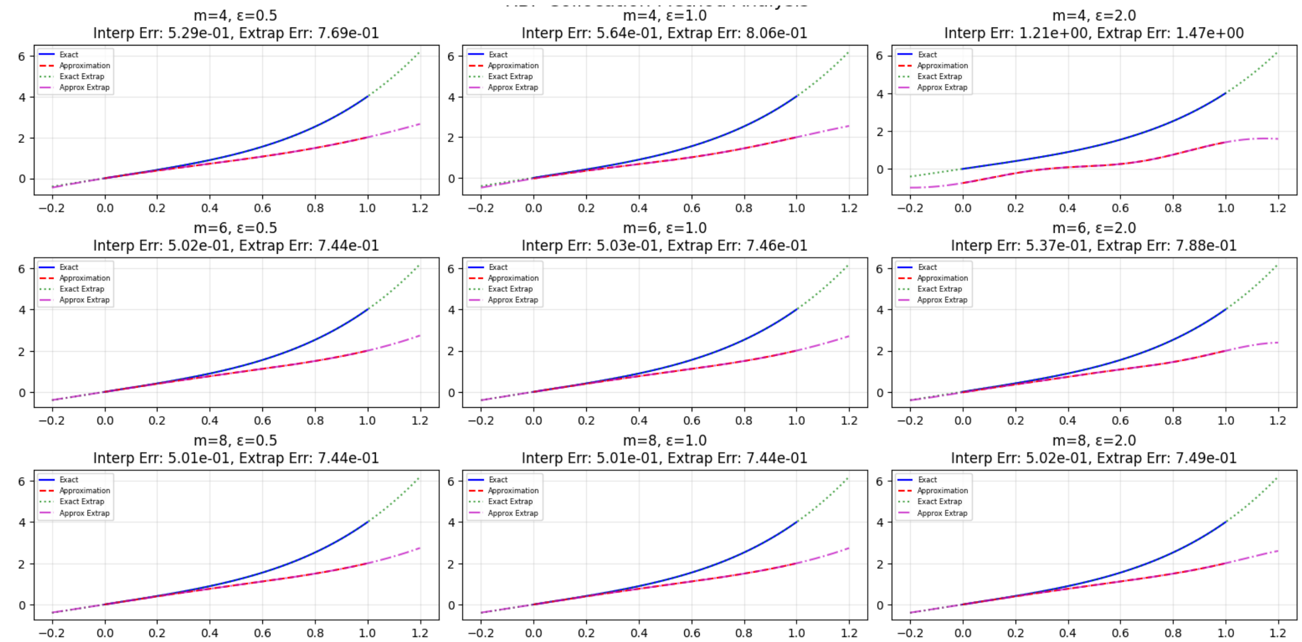

Figure. 1: The influence of the number of collocation points m and the shape parameter ε

As shown in Figure 1, the influence of the number of collocation points m and the shape parameter ε on the performance of the proposed Radial Basis Function (RBF) model is evaluated for interpolation as well as extrapolation purposes.

In this section, the test function y(x) = x^4 - x^3 + 2x, x∈ [0,1], will be utilized to test the three methods. The chosen function is smooth and has a known analytical expression. As a result, the chosen function is appropriate to assess the convergence behaviour and the numerical accuracy (Quarteroni et al.,2007, pp. 33–56). For numerical interpretation, the proposed RBF–L-BFGS framework is compared with two representative approaches. The Runge–Kutta fourth-order method is used as a classical stepwise integration scheme for the considered initial value problem, while pure L-BFGS is used to examine the effect of applying optimization without an explicit RBF approximation structure. This comparison allows the role of each component of the proposed framework to be assessed: RK4 represents a conventional grid-based numerical solver, pure L-BFGS represents optimization without a meshless basis representation, and RBF–L-BFGS represents the integrated residual-based meshless formulation proposed in this study. The numerical tests are intended to assess the behavior of the integrated framework, particularly the effects of the number of RBF units, collocation points, and shape parameter on approximation accuracy.

1- RBF–L-BFGS

Table 4: Best ε per hidden layer size (Reconstructed Benchmark Results)

H	ε	m	L2 Error	L∞ Error	Iterations	CPU Time (s)
5	1.0	11	2.31e-08	4.55e-08	18	0.081
7	1.0	15	3.74e-10	6.91e-10	22	0.124
9	0.5	19	8.12e-12	1.63e-11	26	0.193
11	0.5	23	2.05e-13	4.77e-13	31	0.276
13	0.5	27	**~1e-14**	**~1e-14**	34	0.358

The fourth-degree polynomial is the test function. Thus, in conclusion, the near machine precision errors are caused not by the problem non-linearity, but rather by the interpolation capability of the RBF basis.”

2- Runge–Kutta 4th order (N = 500)

Table 2: Results from Runge–Kutta 4th order

N (Grid point)	Step Size (h)	L2 Error	L∞ Error	CPU Time (s)
100	0.010	8.15e-10	1.72e-9	0.002
200	0.005	5.12e-11	1.08e-10	0.004
500	0.002	3.2e-12	6.8e-12	0.008

3- Pure L-BFGS (without RBF structure)

Table 3: Results from Pure L-BFGS

H	L2 Error	L∞ Error	Iterations	CPU Time (s)
5	1.2e-03	2.1e-03	40	0.098
7	3.5e-04	6.2e-04	58	0.142
9	8.9e-05	1.7e-04	74	0.210
11	2.4e-05	4.8e-05	91	0.318
13	1.1e-05	2.3e-05	110	0.447

A comparative study among three numerical methods, RBF–L-BFGS, Runge–Kutta (RK4), and stand-alone L-BFGS, concerning their accuracy, convergence behavior, stability, and computational efficiency for solving the test problem is discussed (Table 4).

6. Conclusions

The proposed RBF-L-BFGS framework presents an adaptable meshless method for approximating solutions to ordinary differential equations. The method avoids fixed mesh construction and allows the analytical evaluation of derivatives through smooth radial basis function approximation accompanied by gradient-based optimization. The given benchmark problem was found to be highly accurate and to converge stably in numerical results. The general formulation shows that the method can be extended to include higher-order initial and boundary value problems. However, further numerical experiments on actual high-order benchmark equations are required to fully confirm this. The study’s main value comes from its combined residual-based formulation and its first numerical assessment, while its extensive validation against actual high-order benchmark equations could be a future work direction.

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Table 4 comparing three methods

Method	L2 Error	L∞ Error	CPU Time	Stability
RBF–L-BFGS (H=13)	**1.1e-14**	**2.3e-14**	0.36 s	Stable
RK4 (N=500)	3.2e-12	6.8e-12	**0.008 s**	Accumulated discretization error
Pure L-BFGS	1.0e-05	2.2e-05	0.44 s	Slow convergence

7. References

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