# On Solve Fuzzy Boundary Value Problem

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### Abstract

**The main objective of this paper is to study the pyramidal fuzzy numbers in fuzzy differential equations and introduce a new approach for solving fuzzy boundary value problems. Also, we give an example, where we compare the fuzziness of "pyramidal" solution to that one, which is derived by the extension principle.**

**Keywords:** Fuzzy sets, Boundary Fuzzy differential equation, Boundary value problems, Pyramidal fuzzy numbers

## 1-Introduction:

**The concept of fuzzy sets was introduced initially by Zadeh in 1965, [1]. Since then, this concept is used expensively in fuzzy systems described by fuzzy processes which look as their natural extension into the time domain.**

**The term "fuzzy differential equation" was coined in 1978 by A. Kandel, Wj. Byatt, in [2] they were carefully studied, for example, solution of fuzzy differential equations provide a noteworthy example of time dependent fuzzy sets in [3],[4].it was followed up by dubois and prade [5 ] who used the extension principle in their approach. The study of fuzzy differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis [6] and [7] and set-valued differential equations [8] Initially, the derivative for fuzzy valued mappings was developed by Puri and Ralescu the theory of fuzzy differential equations seems to have split into two independent branches, where the first one relies upon the notion of Hukuhara derivative [9], and the second we define the class of pyramidal fuzzy numbers and offer a new definition of the solution to fuzzy differential equations.**

**In the last few years, many works have been done by several authors in theoretical and applied fields see [10, 11, 12]. A variety of exact, approximate and purely numerical methods are available to find the solution of a fuzzy initial value problem. It is an important problem that we know a differential equation has a unique solution. There are many theorems and reasonable conditions for this aim. The shooting method can approximate the unique solution for a linear fuzzy boundary value differential equations see [13, 14].**

**In this paper, the concept of pyramidal fuzzy numbers have been used to solve differential equations and then extended to solve fuzzy boundary value problems (FBVP)**

## 2-Preliminaries

**In this section, several basic concepts of fuzzy system, fuzzy differential equations and function of matrices will be recalled; we start with the obvious definition of fuzzy differential equation.**

**Definition (2.1) [15]:**

A fuzzy set $\widetilde{A}$ on the set of real numbers R is convex if and only if:

$M_{\widetilde{A}}(\lambda x_1 + (1-\lambda)\, x_2) \geq \text{Min}\,\{M_{\widetilde{A}}(x_1), M_{\widetilde{A}}(x_2)\}$… (2.1)

for all $x_1, x_2 \in$ R and all $\lambda \in [0, 1]$.

**Remark (2.2):**

A fuzzy number is a function u: R→ [0, 1] satisfying the following properties:

1-u is normal, i.e. there exists a unique $x_0 \in$ R with $u(x_0) = 1$.

2-u is convex fuzzy set.

3- u is upper semi continuous on R.

4-The support of A is compact.

The family of fuzzy numbers which will be denoted by E and for $0 < r < 1$, $[u]^r$ denotes $\{x \in R: u(x) \geq r\}$ which is called the r-level set and $[u]^0$ denotes $\{x \in R: u(x) > 0\}$which is called the support of the fuzzy number u it should be noted that for any $0 \leq r \leq 1$, $\lambda[u]^r$ is abounded closed interval for u, r∈ E and $\lambda \in$R , the sum u+v and the product $\lambda$u can be defined by :

$[u + v]^r = [u]^r + [v]^r$

$[\lambda\, u]^r = [u]^r$, $r \in [0, 1]$

Where $[u]^r + [v]^r$ means the addition of two intervals of R and $\lambda[u]^r$ means the product between a scalar and a subset of R.

Arithmetic operation of arbitrary fuzzy numbers u= $(\underline{u}(r), \overline{u}(r))$, v= $(\underline{v}(r), \overline{v}(r))$ and $\lambda \in$ R can be defined as:

1-u=v if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$

2-u+ v = $(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$

3-u- v = $(\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r))$ (2.2)

$$4\text{-}\lambda\, u = \begin{cases} \left(\lambda\, \underline{u}(r),\ \lambda\, \overline{u}(r)\right), & \text{if } \lambda \geq 0 \\ \left(\lambda\, \overline{u}(r), \lambda\, \underline{u}(r)\right), & \text{if } \lambda < 0 \end{cases}$$

Note that the crisp number $\sigma$ *is s*imply represented by $\underline{u}(r) = \overline{u}(r) = \sigma$ which is obtained by letting r=1.

In the following, we recall some definitions concerning fuzzy differential equations:

**Definition (2.3) [16]:**

**Let $P_k(R^n)$ denote the family of all non-empty compact convex subsets of $R^n$ and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two non-empty and bounded subsets of $R^n$, then the distance between A and B is defined by the following Housdorff metric:**

$$d(A,B)=\max\{\sup_{a\in A}\inf_{b\in B} d(a,b),\ \sup_{b\in B}\inf_{a\in A} d(b,a)\} \qquad (2.3)$$

**Where d (a, b) denotes the usual distance function in $R^n$.**

**Now, we denote $E^n$ = {u: $R^n \to [0, 1]$ u satisfies (1)-(4) above.**

**<u>Definition (2.4) [17]:</u>**

**A mapping f: T→ E for the interval $T \subseteq R$ is called a fuzzy process. Therefore, its r-level set can be written as follows:**

$$[f(t)]^r=[f_-^r(t), f_+^r(t)]^r,\ t \in T,\ r \in [0,1] \qquad (2.4)$$

**<u>Definition (2.5)[18]:</u>**

**A function $F_\alpha$: $E^n \to P_k(R^n)$ is called Hukuhara differentiable at a point $t_0 \in R^n$ if for h>0 sufficiently small, we have:**

$$\acute{F}_\alpha(t_0) = \lim_{h\to 0^+}\frac{F_\alpha(t_0+h)-F_\alpha(t_0)}{h}$$

$$= \lim_{h\to 0^-}\frac{F_\alpha(t_0)-F_\alpha(t_0-h)}{h}$$

$$= \lim_{h\to 0}\frac{F_\alpha(t_0+h)-F_\alpha(t_0)}{h} \qquad (2.5)$$

**where the limits of Hukuhara derivative are taken in the metric space $(P_k(R^n),d)$, and$F_\alpha(t_0+h)$-$F_\alpha(t_0)$=( $\overline{a}$-$\overline{b}$ ,$\underline{a}$ -$\underline{b}$ ).**

**<u>Definition (2.6)[19]:</u>**

**A mapping F: T→$E^n$ is called differentiable at $t_0 \in T$, if for any $\alpha \in [0, 1]$, the set valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at a point $t_0$ with D $F_\alpha(t_0)$ and the family { D $F_\alpha(t_0)$| $\alpha \in [0,1]$} define a fuzzy number $F'(t_0) \in E^n$, which is called the differentiation of F at $t_0$.**

**If F: T →$E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the fuzzy derivative of F ($t_0$)**

**at the point $t_0$.**

## <u>3. Solutions of Pyramidal Fuzzy Numbers</u>

**The pyramidal number is one of the fundamental aspects in differential equations, in general and of fuzzy differential equations in particular. Therefore, several approaches are proposed to study this subject.**

**Hence, in this section, we will give one of such approaches as a theorem. Also, we will set and present some of the basic ideas for the construction and proof of the pyramidal fuzzification.**

**<u>Definition (3.1)[20]:</u>**

The fuzzy number $x \in E^n$ is called pyramidal if its $\alpha$-level sets are n-dimensional rectangles for $0 \le \alpha \le 1$.

## The Pyramidal Method For Solving Fuzzy Differential Equations (3.2):

There exists a fuzzy process x: $[0,T] \to E^n$ ,such that

$$[x(t)]^{\alpha} = x_k^n \; [g_{1\alpha}^{k}(t), g_{2\alpha}^{k}(t)] \qquad (3.1)$$

Where X denotes the usual set theoretical Cartesian product

Proof:see[20].

## Pyramidal Fuzzification For Solving Linear Fuzzy Systems(3.3):

In this section, if the function $z(t, x_0)$ from the preceding section may be given in a closed from, then a more direct method of fuzzification of the crisp solution may be proposed and used later to solve FBVP'S.

## Theorem

Consider an initial value problem for the linear system

$$X(t) = \begin{pmatrix} x1 \\ x2 \\ \vdots \\ xn \end{pmatrix} = e^{At} \quad \text{Where A (t) is an } n \times n \text{ matrix}$$

$$e^{At} = I + At + \frac{A^2}{2!} t^2 + \ldots \qquad (4.1)$$

Then the fuzzy process x (t) satisfies

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + a(t) \\ x(t_0) &= x_0 \end{aligned}\right\} \qquad (4.2)$$

Where $\dot{x}(t) = \frac{dx(t)}{dt}$, will be solved.

There it can be obtained

$$\tilde{x}(t) = (E_{t_0}^{t}\ \tilde{A})\ \tilde{x}_0 + \int_{t_0}^{t} (E_{t_0}^{t-\mathrm{T}}\ \tilde{A})\ \tilde{a}(\tau) d\tau \qquad (4.3)$$

Proof see [20].

## 4-Solution of Fuzzy Boundary Value Problems

Differential equations which are given with fuzzy conditions given at two or more points of the domain of definition are called FBVP'S. We consider non fuzzy differential equations of order two with boundary fuzzy conditions at the end points a, b.

$$y'' = f(x, y, y'),\ a \le x \le b,\ y(a) = \alpha \ \text{ and } y(b) = \beta \ldots\ldots (1)$$

under what conditions a boundary value problem has a solution or has a unique solution.

## Existence and uniqueness (4.1) [11,16]

Suppose that f is continuous on the set

$$D = \{(x, y, y'); a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$$

**and the partial derivatives $f_y$ and $f_{y'}$ are also continuous on D. if**

**1) $f_y(x, y, y') > 0$, for all (x, y, y′) in D, and**

**2) there exists a constant M such that**

$$|f_{y'}(x, y, y')| \le M \text{ for all } (x, y, y') \text{ in } D,$$

**then the boundary value problem(1) has a unique solution.**

## Example (4.2)

**Consider the following boundary value problem:**

**$y'' + e^{-xy} + \sin(y') = 0$, $1 \le x \le 2$, $y(1) = y(2) = 0$.**

**Determine if the boundary value problem has a unique solution.**

**Rewrite $y'' = -e^{-xy} - \sin(y')$. So $f(x, y, y') = -e^{-xy} - \sin(y')$**

**Check conditions:**

**$f(x, y, y') = -e^{-xy} - \sin(y')$, $f_y(x, y, y') = xe^{-xy}$, and $f_{y'}(x, y, y') = -\cos(y')$ are continuous on**

. $$D = \{(x, y, y'); 1 \le x \le 2, -\infty < y < \infty, -\infty < y' < \infty\}$$

**(i) $f_y(x, y, y') = xe^{-xy} > 0$ on D.**

**(ii) $|f_{y'}(x, y, y')| = |-\cos(y')| \le 1 = M$.**

**So, the boundary value problem has a unique solution in D.**

## Example (4.3)

**Consider the linear boundary value problem:**

$$y'' = p(x)y' + q(x)y + r(x), \; a \le x \le b, \; y(a) = \alpha \text{ and } y(b) = \beta.$$

**Under what condition(s) a linear BVP has a unique solution?**

**$f(x, y, y') = p(x)y' + q(x)y + r(x)$, $f_y(x, y, y') = q(x)$, $f_{y'}(x, y, y') = p(x)$ are continuous on D if p(x),q(x) and r(x) are continuous for $a \le x \le b$**

**a. $f_y(x, y, y') = q(x) > 0$ for $a \le x \le b$.**

**b. Since $f_{y'}$ is continuous on a, b, $f_{y'}$ is bounded.**

**So, if p(x), q(x) and r(x) are continuous for $a \le x \le b$, and $q(x) > 0$ for $a \le x \le b$, then the boundary value problem has a unique solution.**

**The existence of a unique solution for fuzzy initial value problems will relate by the shooting method of the existence of a unique solution for fuzzy boundary value problems is studied depending on the initial fuzzy value problem.**

## 5. The Shooting Method for Solving Fuzzy Linear BVP's [13, 14]:

**The shooting method for solving fuzzy linear differential equation is based on the replacement of the fuzzy boundary value problem by its two related fuzzy initial value problems, as it is in usual case for solving non-fuzzy boundary value problems.**

**Now, consider the linear second order fuzzy boundary value problem:**

$$y'' = p(x)y' + q(x)y + r(x), \; a \le x \le b \quad \text{.......(5.1)}$$

**with fuzzy boundary conditions:**

$$y(a) = \tilde{\alpha},\ y(b) = \tilde{\beta} \qquad \text{..........(5.2)}$$

**where $\tilde{\alpha}$ and $\tilde{\beta}$ are to pyramidal fuzzy .the differential equation**

**(i) p(x), q(x) and r(x) are continuous on [a, b].**

**(ii) q(x) > 0 on [a, b].**

**Hence, the related two fuzzy initial value problems are given by:**

$$u'' = p(x)u' + q(x)u,\ a \le x \le b,\ u(a) = \tilde{0},\ u'(a) = \tilde{1} \qquad \text{..........(5.3)}$$

**and**

$$v'' = p(x)v' + q(x)v + r(x),\ a \le x \le b,\ v(a) = \tilde{\alpha},\ v'(a) = \tilde{0} \qquad \text{..........(5.4)}$$

**To find the solutions of the fuzzy initial value problems (5.3) and (5.4), respectively, the α-level equations related to these fuzzy differential equations are evaluated, which are:**

$$[u'']_\alpha = p(x)[u']_\alpha + q(x)[u]_\alpha,\ [u(a)]_\alpha = \tilde{0}_\alpha,\ [u'(a)]_\alpha = \tilde{1}_\alpha \qquad \text{..........(5.5)}$$

**and**

$$[v'']_\alpha = p(x)[v']_\alpha + q(x)[v]_\alpha + r(x),\ [v(a)]_\alpha = \tilde{\alpha}_\alpha,\ [v'(a)]_\alpha = \tilde{0}_\alpha \qquad \text{..........(5.6)}$$

**We can check that $\underline{y}(x)$ is indeed the solution of the original fuzzy BVP, since:**

$$\underline{y}'(x) = \underline{v}'(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\underline{u}'(x)$$

**and**

$$\underline{y}''(x) = \underline{v}''(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\underline{u}''(x)$$

**So:**

$$\underline{y}''(x) = p(x)\underline{v}'(x) + q(x)\underline{v}(x) + r(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}(p(x)\underline{u}'(x) + q(x)\underline{u}(x))$$

$$= p(x)\{\underline{v}'(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\underline{u}'(x)\} + q(x)\{\underline{v}(x) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\underline{u}(x)\} + r(x)$$

$$= p(x)\underline{y}'(x) + q(x)\underline{y}(x) + r(x)$$

**Moreover:**

$$\underline{y}(a) = \underline{x_1(\theta, t)} = \underline{v}(a) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\underline{u}(a) \qquad , \quad 0 \le \theta \le 2\pi$$

$$= \tilde{\alpha} + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)} \times 0 = \tilde{\alpha}$$

**and**

$$\underline{y}\,(\mathbf{b}) = \underline{\mathbf{x_1}(\boldsymbol{\theta},\mathbf{t})} = \underline{v}\,(\mathbf{b}) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}\,(\mathbf{b})$$

$$= \underline{v}\,(\mathbf{b}) + \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}\,\underline{u}(b)$$

$$= \underline{v}\,(\mathbf{b}) + \tilde{\beta} - \underline{v}\,(\mathbf{b}) = \tilde{\beta}$$

**Hence, $\underline{y}$ (x) is the unique solution to the linear BVP, provided, of course, that $\underline{u}$ (b) ≠ 0.**

**Similarly, as in upper case, we have:**

$$\bar{y}(x) = \overline{\mathbf{x_2}(\boldsymbol{\theta},\mathbf{t})} = \bar{v}\,(x) + \frac{\tilde{\beta} - \bar{v}(b)}{\bar{u}(b)}\,\bar{u}\,(x) \qquad , \qquad 0 \le \boldsymbol{\theta} \le 2\boldsymbol{\pi}$$

**Next, an illustrative example for solving FBVP analytically will be considered.**

**<u>Example (5.1):</u>**

**To solve the nonlinear fuzzy boundary value problem using the shooting method, where:**
**$y'' = 2yy'$, $y(0) = -\tilde{1}$, $y(\frac{\pi}{2}) = -\tilde{1}$, $x \in [0, \frac{\pi}{2}]$**

**Hence the linearized system evaluated at the point $[\frac{1}{2}, 0]$ is given by:**

**Let** $\tilde{y}_1' = \tilde{y}_2$

$\tilde{y}_2' = 2\,\tilde{y}_1\tilde{y}_2$

**Hence in matrix form:**

$$\begin{bmatrix}\tilde{y}_1'\\ \tilde{y}_2'\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}, \qquad \tilde{y}_1(0) = -\tilde{1} \quad \tilde{y}_2((\tfrac{\pi}{2}) = -\tilde{1}$$

$$\tilde{\boldsymbol{\beta}} = -\tilde{1} \qquad \tilde{\boldsymbol{\alpha}} = \tilde{1} \qquad x \in [0, (\tfrac{\pi}{2}]$$

**In order to solve the above fuzzy BVP consider the first non-fuzzy BVP:**

$$\begin{bmatrix}\tilde{u}_1'\\ \tilde{u}_2'\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\end{bmatrix}, \qquad \tilde{u}_1(0) = \tilde{0}\,,\ \tilde{u}_2(0) = \tilde{1} \quad ,$$

**and hence the eigenvalues of A are given by $\lambda_1 = 0$ , $\lambda_2 = 1$**

**Therefore to find the corresponding eigenvectors**

**Let Au= $\boldsymbol{\lambda}$u**

**Then**

$$\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\end{bmatrix} = 0\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\end{bmatrix}$$

**This implies**

**$\tilde{u}_2 = 0\,\tilde{u}_1 \Rightarrow \tilde{u}_1 = r$, r=1**

$\tilde{u}_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\tilde{u}_2=\begin{bmatrix}1\\0\end{bmatrix}$

Hence

p=$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$, $e^{At}$=p$e^{at}$p$^{-1}$

p$^{-1}$ = $\begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}$

Therefore

e$^{at}$= diag ($e^{at}$)

And so

$$\tilde{u}(t)=\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}e^{0t} & 0\\0 & e^{t}\end{bmatrix}\begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}\tilde{u}_{01}\\\tilde{u}_{02}\end{bmatrix}$$

$$=\begin{bmatrix}1 & e^{t}\\0 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}\tilde{u}_{01}\\\tilde{u}_{02}\end{bmatrix}$$

$$=\begin{bmatrix}0 & -1+e^{t}\\0 & 0\end{bmatrix}\begin{bmatrix}\tilde{u}_{01}\\\tilde{u}_{02}\end{bmatrix}$$

Hence

$\underline{\tilde{u}}_1(t) = 0\,\tilde{u}_{01}$ $\quad\overline{\tilde{u}}_1(t) =-1 + e^{t}\,\tilde{u}_{02}$

$\underline{\tilde{u}}_2(t) = 0\,\tilde{u}_{01}$ $\quad\overline{\tilde{u}}_2(t) = 0\,\tilde{u}_{02}$

$\underline{\tilde{u}}_{01}= -\sqrt{1-\alpha}$ $\quad\overline{\tilde{u}}_{01} =\sqrt{1-\alpha}$ $\quad,\alpha=1$

$\underline{\tilde{u}}_{02}= 1-\sqrt{1-\alpha}$ $\quad\overline{\tilde{u}}_{02}=1+\sqrt{1-\alpha}$

Now, consider the second nonfuzzy BVP:

$$\begin{bmatrix}\tilde{v}_1'\\\tilde{v}_2'\end{bmatrix}=\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}\tilde{v}_1\\\tilde{v}_2\end{bmatrix}, \quad \tilde{v}_1(0) = \tilde{1} \ ,\tilde{v}_2(0) = \tilde{0}$$

and hence the eigenvalues of A are given by $\lambda_1 = 0$ , $\lambda_2$= 1

Therefore to find the corresponding eigenvectors

Let Av= $\boldsymbol{\lambda}$v

Then

$$\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}\tilde{v}_1\\\tilde{v}_2\end{bmatrix}=1\begin{bmatrix}\tilde{v}_1\\\tilde{v}_2\end{bmatrix}$$

this implies

$\tilde{v}_2=\tilde{v}_1 \Rightarrow \tilde{v}_1$=r, r=1

$\tilde{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\tilde{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$

Hence

p=$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$, $e^{At}$=p$e^{at}$p$^{-1}$, where

$p^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

**Therefore**

$e^{at} = \text{diag}\,(e^{at})$

**and so**

$$\tilde{v}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{v}_{01} \\ \tilde{v}_{02} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^t \\ e^t & e^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{v}_{01} \\ \tilde{v}_{02} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_{01} \\ \tilde{v}_{02} \end{bmatrix}$$

**Hence:**

$\underline{\tilde{v}}_1(t) = -\cos t\, \tilde{v}_{01} \qquad \overline{\tilde{v}}_1(t) = -\sin t\, \tilde{v}_{02}$

$\underline{\tilde{v}}_2(t) = \sin t\, \tilde{v}_{01} \qquad \overline{\tilde{v}}_2(t) = -\cos t\, \tilde{v}_{02}$

$\underline{\tilde{v}}_{01} = 1 - \sqrt{1-\alpha} \qquad \overline{\tilde{v}}_{01} = 1 + \sqrt{1-\alpha}) \qquad , \ \alpha = 1$

$\underline{\tilde{v}}_{02} = -\sqrt{1-\alpha} \qquad \overline{\tilde{v}}_{02} = \sqrt{1-\alpha}$

**Now** $\underline{\lambda} = \dfrac{\overline{\beta} - \underline{\tilde{v}}_1(t)}{\underline{\tilde{u}}_1(t)} = \dfrac{-1 - 0\tilde{v}01}{0\tilde{u}01}$

$\overline{\lambda} = \dfrac{\overline{\beta} - \overline{\tilde{v}}_1(t)}{\overline{\tilde{u}}_1(t)} = \dfrac{-1 - 0\tilde{v}02}{-1 + e^t\, \tilde{u}02}$

**Since the general solution of fuzzy boundary value problem using the shooting method is given by:**
$\underline{y}(t) = \underline{v}(t) + \lambda\, \underline{u}(t), \ \overline{y}(t) = \overline{v}(t) + \lambda\, \overline{u}(t)$

**The results could be compared with the crisp solution at $\alpha$=1, and for t =$\frac{\pi}{2}$, to given**

$\underline{y}(\frac{\pi}{2}) = \underline{v}(\frac{\pi}{2}) + \lambda\, \underline{u}(\frac{\pi}{2}) = -1.$

$\overline{y}\left(\frac{\pi}{2}\right) = \overline{v}(\frac{\pi}{2}) + \lambda\, \overline{u}(\frac{\pi}{2}) = -1.$

**Where the crisp solution at $t = \frac{\pi}{2}$ is given by:**

$y(\frac{\pi}{2}) = -1.$

**Also, we can make a comparison between the crisp solution and fuzzy solution with $\alpha$=1 as in the following Fig. (1)**

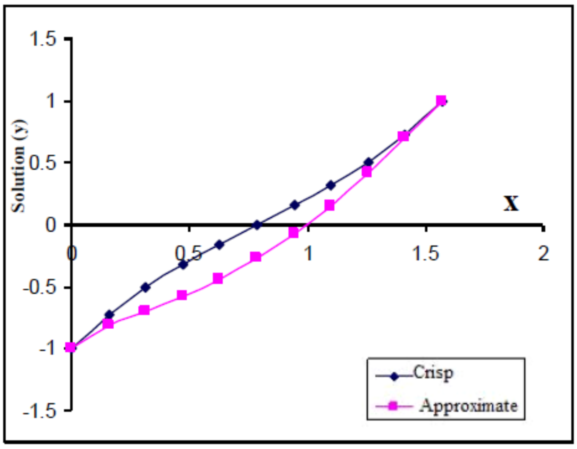


***Fig. (1) : A comparison between crisp and fuzzy solution with , $\alpha$=1***

**Example (5.2):**

**Consider the second order fuzzy differential equation of boundary value problem:**

$y'' = -y$, $y(0) = \tilde{1}$, $y(1) = -\tilde{1}$, $x \in [0, 1]$

**Let** $\tilde{y}_1' = \tilde{y}_2$

$\tilde{y}_2' = -\tilde{y}_1$

**Hence in matrix form:**

$$\begin{bmatrix} \tilde{y}_1' \\ \tilde{y}_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}, \quad \tilde{y}_1(0) = \tilde{1} \quad \tilde{y}_2(1) = -\tilde{1}$$

$$\tilde{\beta} = -\tilde{1} \quad \tilde{\alpha} = \tilde{1} \qquad x \in [0, 1]$$

**In order to solve the above fuzzy BVP consider the first non-fuzzy BVP:**

$$\begin{bmatrix} \tilde{u}_1' \\ \tilde{u}_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \quad \tilde{u}_1(0) = \tilde{0}, \ \tilde{u}_2(0) = \tilde{1} \ ,$$

**and hence the eigenvalues of A are given by** $\lambda_1 = i$, $\lambda_2 = -i$

**Therefore to find the corresponding eigenvectors**

**Let** $Au = \lambda u$

**Then**

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = i \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$$

**This implies**

$\tilde{u}_2 = i\tilde{u}_1 \Rightarrow \tilde{u}_1 = r, \ r = 1$

$$\tilde{u} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Hence**

$$p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ e^{At} = p e^{jt} p^{-1}$$

$$p^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

**Therefore**

$$e^{jt} = e^{at} \begin{bmatrix} \cos bjt & -\sin bjt \\ \sin bjt & \cos bjt \end{bmatrix}$$

**And so**

$$\tilde{u}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_{01} \\ \tilde{u}_{02} \end{bmatrix}$$

$= \begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_{01} \\ \tilde{u}_{02} \end{bmatrix}$

$= \begin{bmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} \tilde{u}_{01} \\ \tilde{u}_{02} \end{bmatrix}$

**Hence**

$\underline{\tilde{u}}_1(t) = -\cos t\, \tilde{u}_{01}$ $\quad \overline{\tilde{u}}_1(t) = -\sin t\, \tilde{u}_{02}$

$\underline{\tilde{u}}_2(t) = \sin t\, \tilde{u}_{01}$ $\quad \overline{\tilde{u}}_2(t) = -\cos t\, \tilde{u}_{02}$

$\underline{\tilde{u}}_{01} = -\sqrt{1-\alpha}$ $\quad \overline{\tilde{u}}_{01} = \sqrt{1-\alpha}$ $\quad ,\alpha=1$

$\underline{\tilde{u}}_{02} = 1-\sqrt{1-\alpha}$ $\quad \overline{\tilde{u}}_{02} = 1+\sqrt{1-\alpha}$

**Now, consider the second nonfuzzy BVP:**

$\begin{bmatrix} \tilde{v}_1' \\ \tilde{v}_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}, \quad \tilde{v}_1(0) = \tilde{1} \ , \tilde{v}_2(0) = \tilde{0}$

**and hence the eigenvalues of A are given by $\lambda_1 = i$ , $\lambda_2 = -i$**

**Therefore to find the corresponding eigenvectors**

**Let Av= $\lambda$v**

**Then**

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = i \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$

**this implies**

$\tilde{v}_2 = i\tilde{v}_1 \Rightarrow \tilde{v}_1 = r,\ r=1$

$\tilde{v} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

**Hence**

$p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $e^{At} = pe^{jt}p^{-1}$, **where**

$p^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

**Therefore**

$e^{jt} = e^{at} \begin{bmatrix} \cos bjt & -\sin bjt \\ \sin bjt & \cos bjt \end{bmatrix}$

**and so**

$\tilde{v}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_{01} \\ \tilde{v}_{02} \end{bmatrix}$

$= \begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_{01} \\ \tilde{v}_{02} \end{bmatrix}$

$= \begin{bmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} \tilde{v}_{01} \\ \tilde{v}_{02} \end{bmatrix}$

**Hence:**

$\underline{\tilde{v}}_1(t) = -\cos t\, \tilde{v}_{01}$ $\quad \overline{\tilde{v}}_1(t) = -\sin t\, \tilde{v}_{02}$

$\underset{\sim}{v}_2$ (t)=sint $\underset{\sim}{v}_{01}$ $\quad$ $\overline{\tilde{v}}_2$ (t) = -cost $\tilde{v}_{02}$

$\underset{\sim}{v}_{01}$= 1-$\sqrt{1-\alpha}$ $\quad$ $\overline{\tilde{v}}_{01}$=$1+\sqrt{1-\alpha})$ $\quad$ , $\alpha$=1

$\underset{\sim}{v}_{02}$= -$\sqrt{1-\alpha}$ $\quad$ $\overline{\tilde{v}}_{02}$ = $\sqrt{1-\alpha}$

Now $\underline{\lambda}$= $\frac{\overline{\beta}-\underset{\sim}{v}_1(t)}{\underset{\sim}{u}_1(t)}$=$\frac{-1+\text{cost }\tilde{v}01}{-\text{cost }\tilde{u}01}$

$\overline{\lambda}$=$\frac{\overline{\beta}-\overline{\tilde{v}}_1(t)}{\overline{\tilde{u}}_1(t)}$ =$\frac{-1+\text{sint }\tilde{v}02}{-\text{sint }\tilde{u}02}$

he results could be checked by comparing them with the crisp solution at $\alpha$=1 , and for $t$ = 1, we have

$\underline{\text{y}}$ (t) = $\underline{\text{v}}$ (t) + $\lambda$ $\underline{\text{u}}$ (t) = -0.9992.

$\overline{y}(t)$ = $\overline{\text{v}}(t)$ + $\lambda$ $\overline{\text{u}}$ $(t)$ = -0.9992.

Where the crisp solution at $t$ = 1 is given by y(1) = -1

Also, we can make a comparison between the crisp solution and fuzzy solution with $\alpha$=1 as it is illustrative in Fig. (2)

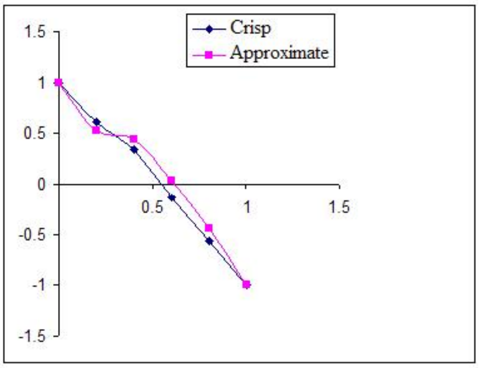


***Fig. (2): A comparison between the crisp solution and fuzzy solution at $\alpha$=1.***

As a boundary value for the FBVP of $\theta$ we take a fuzzy number $x_0 \in E^n$, such that

$[x_0]^\alpha$={$(x_1^0, x_2^0) \in R^2 | (x_1^0 - x_1^{00})$+ $(x_2^0 - x_2^{00})^2 \leq r_0^2\,(1-\alpha)^2$}

$x_1^{00} \in$ R, $x_2^{00} \in$ R, $r_0 > 0$.

Using Nguyen's theorem [20], we see that the $\alpha$ level sets of the fuzzy solution u (t) will be convex compact sets in $R^2$, their boundaries having the following parametric representation:

$x^2 + y^2 = r^2[1-M_{x_0}(x, y)]^2$ $\quad$ ,r > 0

$(x_1^0 - x_1^{00})^2 + (x_2^0 - x_2^{00})^2 \leq r_0^2\,(1-\alpha)^2$

$x_1^0 - x_1^{00}$= r $(\alpha)$ $cos\theta$

$x_2^0 - x_2^{00}$=r $(\alpha)$ $sin\theta$

$x_1^0$=r $(\alpha)$ $cos\theta + x_1^{00}$

$x_2^0$=r $(\alpha)$ $sin\theta + x_2^{00}$

And hence the general solution of the FBVP using the shooting method is given by:

$\underline{x_1(\theta, t)}$ = $\underline{v_1}$(t) +$\underline{\lambda}$ $\underline{u_1}$(t), $\overline{x_2(\theta, t)}$ = $\overline{v_1}$(t) +$\overline{\lambda}$ $\overline{u_1}$(t)

$\underline{x_1(\theta, t)}$ =-cost(r ($\alpha$) $cos\theta$+$x_1^{00}$)- sint(r ($\alpha$) $sin\theta$+$x_2^{00}$)

$\overline{x_2(\theta, t)}$ =sint(r ($\alpha$) $cos\theta$+$x_1^{00}$)- cost(r ($\alpha$) $sin\theta$+$x_2^{00}$)

$[x_0]^{\alpha}=\{(x_1^0, x_2^0) \in R^2 | r^2(\alpha)\, cos^2\theta + r^2(\alpha)\, sin^2\theta$

$$\leq r^2(\alpha)\,(cos^2\theta + sin^2\theta)$$

$$\leq r^2(\alpha) \quad \leq r_0^2\,(1-\alpha)^2$$

$r(\alpha) = \quad r_0(1-\alpha)$ $\quad ,0\leq \alpha \leq 1 \quad , \qquad 0 \leq \theta \leq 2\pi$

Now, consider the fuzzy boundary values

$x_1^0 = -1,\ x_2^0 = 1\ , x_1^{00} = x_2^{00} = 0,\ r_0$=2 and $\alpha$ ∈[0, 1] .for t = 0,0.1,0.2,…,1 the approximate solution for $\alpha$=0, 0.2, 0.4, 0.6, 0.8 are obtained in tables 1 to 5.

Table 1: the approximate solution $\underline{x_1(\theta, t)}$ and $\overline{x_2(\theta, t)}$

| $\alpha = 0.2$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| T | θ=0° | | θ=90° | | θ=180° | | θ=270° | | θ=360° | |
| | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ |
| 0 | 1.6 | 0 | -1.6 | 0 | -1.6 | 0 | 1.6 | 0 | 1.6 | 0 |
| 0.1 | 1.598 | 0.003 | -1.598 | 0.003 | -1.598 | 0.003 | 1.598 | 0.003 | 1.598 | 0.003 |
| 0.2 | 1.598 | 0.005 | -1.598 | 0.005 | -1.598 | 0.005 | 1.598 | 0.005 | 1.598 | 0.005 |
| 0.3 | 1.598 | 0.008 | -1.598 | 0.008 | -1.598 | 0.008 | 1.598 | 0.008 | 1.598 | 0.008 |
| 0.4 | 1.598 | 0.011 | -1.598 | 0.011 | -1.598 | 0.011 | 1.598 | 0.011 | 1.598 | 0.011 |
| 0.5 | 1.598 | 0.014 | -1.598 | 0.014 | -1.598 | 0.014 | 1.598 | 0.014 | 1.598 | 0.014 |
| 0.6 | 1.598 | 0.016 | -1.598 | 0.016 | -1.598 | 0.016 | 1.598 | 0.016 | 1.598 | 0.016 |
| 0.7 | 1.598 | 0.019 | -1.598 | 0.019 | -1.598 | 0.019 | 1.598 | 0.019 | 1.598 | 0.019 |
| 0.8 | 1.598 | 0.022 | -1.598 | 0.022 | -1.598 | 0.022 | 1.598 | 0.022 | 1.598 | 0.022 |
| 0.9 | 1.598 | 0.026 | -1.598 | 0.026 | -1.598 | 0.026 | 1.598 | 0.026 | 1.598 | 0.026 |
| 1 | 1.598 | 0.027 | -1.598 | 0.027 | -1.598 | 0.027 | 1.598 | 0.027 | 1.598 | 0.027 |

**Table 2: the approximate solution ‗(x_1 (θ,t) ) and ¯(x_2 (θ,t)**

| $\alpha = 0.4$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| T | θ=0° | | θ=90° | | θ=180° | | θ=270° | | θ=360° | |
| | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ |
| 0 | 1.2 | 0 | -1.2 | 0 | -1.2 | 0 | 1.2 | 0 | 1.2 | 0 |
| 0.1 | 1.199 | 0.002 | -1.199 | 0.002 | -1.199 | 0.002 | 1.199 | 0.002 | 1.199 | 0.002 |
| 0.2 | 1.199 | 0.004 | -1.199 | 0.004 | -1.199 | 0.004 | 1.199 | 0.004 | 1.199 | 0.004 |
| 0.3 | 1.199 | 0.006 | -1.199 | 0.006 | -1.199 | 0.006 | 1.199 | 0.006 | 1.199 | 0.006 |
| 0.4 | 1.199 | 0.008 | -1.199 | 0.008 | -1.199 | 0.008 | 1.199 | 0.008 | 1.199 | 0.008 |
| 0.5 | 1.199 | 0.010 | -1.199 | 0.010 | -1.199 | 0.010 | 1.199 | 0.010 | 1.199 | 0.010 |
| 0.6 | 1.199 | 0.012 | -1.199 | 0.012 | -1.199 | 0.012 | 1.199 | 0.012 | 1.199 | 0.012 |
| 0.7 | 1.199 | 0.014 | -1.199 | 0.014 | -1.199 | 0.014 | 1.199 | 0.014 | 1.199 | 0.014 |
| 0.8 | 1.199 | 0.017 | -1.199 | 0.017 | -1.199 | 0.017 | 1.199 | 0.017 | 1.199 | 0.017 |
| 0.9 | 1.199 | 0.019 | -1.199 | 0.019 | -1.199 | 0.019 | 1.199 | 0.019 | 1.199 | 0.019 |
| 1 | 1.199 | 0.020 | -1.199 | 0.020 | -1.199 | 0.020 | 1.199 | 0.020 | 1.199 | 0.020 |

**Table 3: the approximate solution $\underline{x_1(\boldsymbol{\theta}, \mathbf{t})}$ and $\overline{x_2(\boldsymbol{\theta}, \mathbf{t})}$**

| $\alpha = 0.6$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| T | θ=0° | | θ=90° | | θ=180° | | θ=270° | | θ=360° | |
| | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ |
| 0 | 0.8 | 0 | -0.8 | 0 | -0.8 | 0 | 0.8 | 0 | 0.8 | 0 |
| 0.1 | 0.799 | 0.002 | -0.799 | 0.002 | -0.799 | 0.002 | 0.799 | 0.002 | 0.799 | 0.002 |
| 0.2 | 0.799 | 0.002 | -0.799 | 0.002 | -0.799 | 0.002 | 0.799 | 0.002 | 0.799 | 0.002 |
| 0.3 | 0.799 | 0.004 | -0.799 | 0.004 | -0.799 | 0.004 | 0.799 | 0.004 | 0.799 | 0.004 |
| 0.4 | 0.799 | 0.007 | -0.799 | 0.007 | -0.799 | 0.007 | 0.799 | 0.007 | 0.799 | 0.007 |
| 0.5 | 0.799 | 0.007 | -0.799 | 0.007 | -0.799 | 0.007 | 0.799 | 0.007 | 0.799 | 0.007 |
| 0.6 | 0.799 | 0.008 | -0.799 | 0.008 | -0.799 | 0.008 | 0.799 | 0.008 | 0.799 | 0.008 |
| 0.7 | 0.799 | 0.001 | -0.799 | 0.001 | -0.799 | 0.001 | 0.799 | 0.001 | 0.799 | 0.001 |
| 0.8 | 0.799 | 0.011 | -0.799 | 0.011 | -0.799 | 0.011 | 0.799 | 0.011 | 0.799 | 0.011 |
| 0.9 | 0.799 | 0.012 | -0.799 | 0.012 | -0.799 | 0.012 | 0.799 | 0.012 | 0.799 | 0.012 |
| 1 | 0.799 | 0.014 | -0.799 | 0.014 | -0.799 | 0.014 | 0799 | 0.014 | 0.799 | 0.014 |

**Table 4: the approximate solution $\underline{x_1(\boldsymbol{\theta}, \mathbf{t})}$ and $\overline{x_2(\boldsymbol{\theta}, \mathbf{t})}$**

| $\alpha = 0.8$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| T | θ=0° | | θ=90° | | θ=180° | | θ=270° | | θ=360° | |
| | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ |
| 0 | 0.4 | 0 | -0.4 | 0 | -0.4 | 0 | 0.4 | 0 | 0.4 | 0 |
| 0.1 | 0.391 | 0.001 | -0.391 | 0.001 | -0.391 | 0.001 | 0.391 | 0.001 | 0.391 | 0.001 |
| 0.2 | 0.391 | 0.001 | -0.391 | 0.001 | -0.391 | 0.001 | 0.391 | 0.001 | 0.391 | 0.001 |
| 0.3 | 0.391 | 0.002 | -0.391 | 0.002 | -0.391 | 0.002 | 0.391 | 0.002 | 0.391 | 0.002 |
| 0.4 | 0.391 | 0.003 | -0.391 | 0.003 | -0.391 | 0.003 | 0.391 | 0.003 | 0.391 | 0.003 |
| 0.5 | 0.391 | 0.004 | -0.391 | 0.004 | -0.391 | 0.004 | 0.391 | 0.004 | 0.391 | 0.004 |
| 0.6 | 0.391 | 0.004 | -0.391 | 0.004 | -0.391 | 0.004 | 0.391 | 0.004 | 0.391 | 0.004 |
| 0.7 | 0.391 | 0.005 | -0.391 | 0.005 | -0.391 | 0.005 | 0.391 | 0.005 | 0.391 | 0.005 |
| 0.8 | 0.391 | 0.006 | -0.391 | 0.006 | -0.391 | 0.006 | 0.391 | 0.006 | 0.391 | 0.006 |
| 0.9 | 0.391 | 0.006 | -0.391 | 0.006 | -0.391 | 0.006 | 0.391 | 0.006 | 0.391 | 0.006 |
| 1 | 0.391 | 0.007 | -0.391 | 0.007 | -0.391 | 0.007 | 0.391 | 0.007 | 0.391 | 0.007 |

Table 5: the approximate solution $\underline{x_1(\theta, t)}$ and $\overline{x_2(\theta, t)}$

| $\alpha = 0$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| T | θ=0° | | θ=90° | | θ=180° | | θ=270° | | θ=360° | |
| | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ | $X_1$ | $X_2$ |
| 0 | 2 | 0 | -2 | 0 | -2 | 0 | 2 | 0 | 2 | 0 |
| 0.1 | 1.998 | 0.004 | -1.998 | 0.004 | -1.998 | 0.004 | 1.998 | 0.004 | 1.998 | 0.004 |
| 0.2 | 1.998 | 0.006 | -1.998 | 0.006 | -1.998 | 0.006 | 1.998 | 0.006 | 1.998 | 0.006 |
| 0.3 | 1.998 | 0.01 | -1.998 | 0.01 | -1.998 | 0.01 | 1.998 | 0.01 | 1.998 | 0.01 |
| 0.4 | 1.998 | 0.014 | -1.998 | 0.014 | -1.998 | 0.014 | 1.998 | 0.014 | 1.998 | 0.014 |
| 0.5 | 1.998 | 0.018 | -1.998 | 0.018 | -1.998 | 0.018 | 1.998 | 0.018 | 1.998 | 0.018 |
| 0.6 | 1.998 | 0.02 | -1.998 | 0.02 | -1.998 | 0.02 | 1.998 | 0.02 | 1.998 | 0.02 |
| 0.7 | 1.998 | 0.024 | -1.998 | 0.024 | -1.998 | 0.024 | 1.998 | 0.024 | 1.998 | 0.024 |
| 0.8 | 1.998 | 0.03 | -1.998 | 0.03 | -1.998 | 0.03 | 1.998 | 0.03 | 1.998 | 0.03 |
| 0.9 | 1.998 | 0.032 | -1.998 | 0.032 | -1.998 | 0.032 | 1.998 | 0.032 | 1.998 | 0.032 |
| 1 | 1.998 | 0.034 | -1.998 | 0.034 | -1.998 | 0.034 | 1.998 | 0.034 | 1.998 | 0.034 |

## 6 - Conclusion

In this paper, we applied the pyramidal fuzzy numbers to solve fuzzy boundary value problems under generalized shooting method another application of fuzzy boundary value problems. This method can be extended for an nth order fuzzy boundary value problem. In the future, and following the ideas, we plan to consider the equations $y'(t) = -ty(t)$, or $y'(t) = c_1 y^2(t)+c_2$ with $c_1, c_2$ arbitrary constants.

## References

[1]L. Zadeh, Fuzzy sets, Inform. And control 8(1965)338-353.

[2]A. Kandel, W. J. Byatt, Fuzzy differential equations, in proc. Internet. Conf. on Cybernetics and society, Tokyo- Kyoto, jupon, November 3 -7, 1978, pp. 1213- 1216.

[3]A. Kandel, W.J. Byatt, fuzzy processes, fuzzy sets and systems 4(1980)117-152.

[4]B.P. Lientz, on time dependent fuzzy sets, Inform. Sci. 4(1972)367-376.

[5]D. Dubois, H. Prade, Towards fzzy differential calculus: part3, differentiation, fuzzy Sets and Systems 8(1982)225-233.

[6]P. Diamond, P. Kloeden, Metric Space of Fuzzy Sets:Theory and Application, World Scientific, Singapore, 1994.

[7]C.V. Negoita, D. A. Ralescu, Applications of Fuzzy Sets to System Analysis, Birkhauser, Basel, 1975.

[8]V. Lakshmikantham,T. Gnana Bhaskar, J. Vasundhara Devi, Theory of Set Differential Equations in Metric Spaces, Cambridge Scientific Publishers, Cambridge, 2006.

[9]M.L. Puri, D.A. Ralescu, Differentials of fuzzy function, J. Math. Anal. Appl. 91(1983)552-558.

[10]T. Allahviranloo, E. Ahmad, N. Ahmady, A method for solving N-th order fuzzy differential, International journal of computer mathematics, 2007, Volume86, Issue4, First published 2009, Pages 730-742.

[11]W. Congxin, S.Shiji, Existence theorem to the Cauchy problem of fuzzy differential equations under compactness-type conditions, Inform Sci 108(1998) 123-134.

[12]J. J. Nieto, R. Rodriguez-Lopez Bounded solutions for fuzzy differential and integral equations, Chaos, Solitons and Fractals 27(2006)1376-1386.

[13]L.Jamshid:[1],L.Avazpour[2] "solution of the fuzzy boundary value differential equations under generalized differentiability by shooting method, Journal of Fuzzy set Valued Analysis,Volume 2012,Year 2012,pp.1-19.

[14] Marwa Mohamed, Ismael" the shooting method for solving fuzzy linear boundary value problems "International Journal of Scientific and Engineering Research, volume5, Issue7, July 2014, pp.930-934.

[15]George, J.Klir and Boynan, "fuzzy sets and" fuzzy logic and Application", printichall , Inc.,1995.

[16]Song, S and Wu, C.:"Existence and uniqueness of solutions to the Cauchy problem of fuzzy Differential Equations", fuzzy sets and system, 110, (2000), pp.55-67.

[17]P.E.Kloeden, Remarks on Peano- like theorems for fuzzy differential equations, fuzzy sets and systems 44(1991)161-163.

[18]O.Kaleva, the Cauchy problem for fuzzy differential equations, fuzzy sets and systems 35(1990)389-396.

[19]Bede, Budgal, S.G.: Generalization of the Differentiability of fuzzy-Number-valued functions with Applications to fuzzy Differential Equations", Fuzzy sets and systems, 2004.

[20]D. Vorobier, S.Seikkala,Towords the theory of fuzzy differential equations, fuzzy sets and systems 125(2002)231-237.

# المعادلات التفاضلية الضبابية

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## المستخلص

الهدف الرئيس من هذا البحث هو دراسة الاعداد الضبابية الهرمية في المعادلات التفاضلية الضبابية واستخدام نهج مطور لحل المعادلات التفاضلية الضبابية الحدودية. وايضا اعطاء مثال ،حيث قارنا بين الحلول الضبابية الهرمية للمعادلات التفاضلية الضبابية الحدودية لهذه المسألة والتي اشتقت بواسطة مبدأ التوسيع

الكلمات المفتاحية: مجموعات ضبابي، الحدود المعادلة التفاضلية، مشاكل قيمة الحدود.