Using Laplace Transformation Technique to solve boundary value problems

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Abstract:

In this paper, applying Laplace transform technique will be discussed and to solve partial differential equation with boundary conditions that have significant importance in engineering and physical applications, where two kinds of partial differential equations were solved using these transformations on both sides of the equations then applying the boundary equations to find the general solutions.

Keywords: Laplace transform, Partial Differential Equations, Boundary Value Problems.

الخلاصه

في هذاالبحث سيتم استخدام تحويلات لابلاس في حل المسائل ذات الشروط الحدوديه والتي تظهر في بعض انواع المعادلات التفاضليه الجزئيه ذات الاهميه في التطبيقات الهندسيه والفيزيائيه،حيث تم حل نوعين من المعادلات الجزئيه التفاضليه عبر استخدام تلك التحويلات على طرفي المعادلات ثم تطبيق الشروط الحدوديه لايجاد الحل النهائي.

الكلمات المفتاحية: تحويل لابلاس، المعادلات التفاضلية الجزئية، مشاكل قيمة الحدود

Introduction:

In various areas of science and engineering we might use Laplace transformation technique to solve our problem because it is very powerful mathematical tool [7]. It can solve initial value problem in ordinary differential equation, initial value problem and boundary value problem in partial differential equations.

There is no general method to solve P.D.E [6], but some boundary value problem might be faced, that can be solved using differential transformation [4], and, Laplace transformation method is applied to the time domain [3]. It can solve O.D.E. and P.D.E. because it can transfer O.D.E. to algebraic equation as well as transferring P.D.E. to O.D.E., furthermore, Laplace transformation when facing an infinite domain, it can handle the boundary condition effectively [5].

1. Laplace transformation technique

The immediate Laplace transformation formula for a function $f(t)$ be a function defined for $(0 \leq t \leq \infty)$, then $f(t)$ will have the following Laplace integral:

$$\int_0^\infty f(t)e^{-st}dt,$$

which will be denoted $\mathcal{L}(f(t))$ [1]

1.1 Theorem 1: if $f(t)$ have an exponential order and is piecewise regular on $[0, \infty)$ and exponential order, then for any value of s which greater than the abscissa of convergence of $f(t)$, the integral $\int_0^\infty f(t)e^{-st}dt$ converges, (1).

1.2 Definition: The improper integral $\int_0^\infty F(s,t)dt$ is said to be converge uniformly over a given set S of s values of a given any $\varepsilon > 0$, there exist a number B, depending on ε but not s, such that

$$\left|\int_0^\infty F(s,t)\,dt\right| < \varepsilon$$

for $b > B$ and all $s$in the set S

1.3 Theorem 2 if $f(t)$ is piecewise regular and of exponential order with abscissa of convergence α_0, then for any number $s_0 > \alpha_0$,

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Converges uniformly for all values of s such that $s_0 > \alpha_{0.}$,[1].

2. Some Properties of Laplace Transformation [2]

2.1 $\mathcal{L}$ is a linear transformation where the sum between functions will be:

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]dt = \alpha \int_0^\infty e^{-st} f(t)dt + \beta \int_0^\infty \beta g(t)dt$$

Whenever both integrals converge for $s > c$.

$\mathcal{L}\{\alpha f(t) + \beta g(t)\}dt = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$.

2.2 Theorem: Transform of a Derivative [2]

If $f, f', \dots, f^{(n-1)}$ are continuous on $[0,\infty)$ and are of exponential order and if $f^{(n)}$ is piecewise-continuous on $[0,\infty)$, then

$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - f^{(n-1)}(0)$.

2.3 Example: consider the following problem:

$f_{tt} - v^2 f = 0$

$f(0) = 0$

$f_t(0) = v$

Solution: Apply theorem 2.1

$\mathcal{L}(f_{tt} - v^2 f) = \mathcal{L}(0)$

We know that $\mathcal{L}$ is linear

$\therefore\ \mathcal{L}f_{tt} + v^2\mathcal{L}f = 0$

$s^2F(s) - sf(0) - f_t(0) - v^2F(s) = 0$

Substituting initial conditions

$s^2F(s) - v - v^2F(s) = 0$

$(s^2 - v^2)\, F(s) = v$

$F(s) = \frac{v}{s^2-v^2}$

According to table (1) $\mathcal{L}^{-1}F(s) = \sinh vt$

After solving this example, it can be noticed that the true difficulty regarding the application of Laplace transform is to obtain on inversion criteria.

Table 1: table of Laplace transforms

	$f(x)$	$F(x)$	$a(s > a)$
1	1	$\frac{1}{s}$	0
2	e^{st}	$\frac{1}{s-a}$	a
3	$t^n(n = 1,2,\dots)$	$\frac{n!}{s^{n+1}}$	0
4	$t^n e^{at}(n = 1,2,\dots)$	$\frac{n!}{(s-a)^{n+1}}$	a
5	$\sin kt$	$\frac{k}{s^2+k^2}$	0
6	$\cos kt$	$\frac{s}{s^2+k^2}$	0
7	$\sinh kt$	$\frac{k}{s^2-k^2}$	$\lvert k \rvert$
8	$\cosh kt$	$\frac{s}{s^2-k^2}$	$\lvert k \rvert$
9	$e^{-at}\sin kt$	$\frac{k}{(s+a)^2+}$	$-a$
10	$e^{-at}\cos kt$	$\frac{k}{(s+a)^2+}$	$-a$
11	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2\sqrt{s^3}}$	0
12	$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{s}$	0

Application

Example 3.1: Let the mathematical model of the displacement $f(x,t)$ of point in a string that has a length L at rest with fixed ends, and a force F_0sinhωt., was applied at initial

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} + Fosinh\,\omega t$$

3.1

$$f(0,t) = 0, \quad t > 0$$

3.1.1

$$f(L,t) = 0, \quad t > 0$$

3.1.2

$$f(x,0) = 0, \quad 0 < x < L$$

3.1.3

$$\frac{\partial f}{\partial t} = 0, \quad 0 < x < L$$

3.1.4

To begin, take the Laplace transform equation to obtain

$$s^2F(s) - sf(0) - f_t(0) = c^2\frac{\partial^2}{\partial x^2}\mathcal{L}\{f\} + F_o\frac{\omega}{s^2+\omega^2}$$

By applying the boundary conditions (3.1.1) and (3.1.2), the following equations becomes:

$$s^2F(x,s) = c^2\frac{\partial^2(x,s)}{\partial x^2} + F_o\frac{\omega}{s^2+\omega^2}, \qquad 0 < x < L$$

Or

$$\frac{\partial^2F}{\partial x^2} - \frac{s^2}{c^2}F - f_t(0) = -\frac{F_o\omega}{c^2(s^2+\omega^2)} \qquad 0 < x < L$$

This is now an ordinary differential equation that is subject to the transformed condition:

$$F(0,s) = 0$$

3.2

$$F(L,s) = 0$$

3.3

The homogeneous solution to this equation is

$$F(x,s) = Ae^{\frac{s}{c}x} + Be^{-\frac{s}{c}x}$$

A particular solution can be found by assuming that $F_x x = 0,$. This will give

$$F(x,s) = \frac{F_o\omega}{s^2(s^2-\omega^2)}$$

Therefore, the general solution will be:

$$F(x,s) = Ae^{\frac{s}{c}x} + Be^{-\frac{s}{c}x} + \frac{F_o\omega}{s^2(s^2-\omega^2)}$$

Applying the boundary conditions (3 .2) and (3.3), the following equations are obtained:

$$0 = A + \frac{F_o\omega}{s^2(s^2-\omega^2)}, \qquad 0 = Ae^{\frac{s}{c}x} + Be^{-\frac{s}{c}x} + \frac{F_o\omega}{s^2(s^2-\omega^2)}$$

$$A = -\frac{F_o\omega}{s^2(s^2-\omega^2)}$$

$$B = \frac{F_o\omega}{s^2(s^2-\omega^2)}e^{\frac{2sL}{c}} - \frac{F_o\omega}{s^2(s^2-\omega^2)}$$

Hence,

$$U(x,s) = -\frac{F_o\omega}{s^2(s^2-\omega^2)}e^{\frac{s}{c}x} + \frac{F_o\omega}{s^2(s^2-\omega^2)}e^{-\frac{s}{c}x} \times\left(-1+e^{\frac{s}{c}x}\right)e^{-\frac{s}{c}x} + \frac{F_o\omega}{s^2(s^2-\omega^2)}$$

Example 3.2

Now, consider what might be referred to as one dimensional wave problem:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) + \cos(\pi x), \quad 0 < x < 1, \qquad t > 0$$

$$u(x,0) = 0, \qquad u_t(x,0) = 0$$

$$u(0,t) = 0, \qquad u(1,t) = 0$$

By applying the Laplace transform for the equation, and the conditions must be used, to get

$$\frac{d^2U}{dx^2}(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0) - \frac{\cos \pi x}{s}$$
$$= s^2U(x,s) - \frac{\cos \pi x}{s}$$

The ordinary differential equation, which is non homogeneous with constant coefficient must be solved.

$$\frac{d^2U}{dx^2}(x,s) - s^2U(x,s) = \frac{\cos \pi x}{s}$$

Once again

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

Where $U_h(x,s)$ is the general solution of the homogeneous problem:

$$U_h(x,s) = c_1e^{sx} + c_2e^{-sx}$$

Where $U_p(x,s)$ is particular solution of the non-homogeneous problem:

$$U_p(x,s) = A\cos(\pi x) + B\sin(\pi x)$$

Use the undetermined coefficient method to find A and$B$,where;

$$\frac{d}{dx}U_p(x,s) = -\pi A\sin(\pi x) + \pi B\, cos(\pi x)$$

$$\frac{d^2}{dx^2}U_p(x,s) = -\pi^2 A\cos(\pi x) + \pi^2 B\, sin(\pi x)$$

Therefore

$$\frac{d^2}{dx^2}U_p(x,s) - s^2U_p(x,s) = (-\pi^2 - s^2)[A\cos(\pi x) + B\sin(\pi x)]$$
$$= -\frac{\cos(\pi x)}{s}$$

Which will lead to the following:

$-(s^2 + \pi^2)B = 0$ and $-(s^2 + \pi^2)A = -\frac{1}{s}$, So that

$$B = 0, \quad A = \frac{1}{s(s^2 + \pi^2)}$$

Where;

$$U_p(x, s) = \frac{\cos(\pi x)}{s(s^2 + \pi^2)}$$

And

$$U(x, s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{\cos(\pi x)}{s(s^2 + \pi^2)}$$

Next by applying the BCs to find $c_1$and c_2

$$c_1 = \frac{1}{s(s^2 + \pi^2)} \quad ; \quad c_2 = 0$$

$$\therefore u(x, s) = \frac{e^{sx}}{s(s^2 + \pi^2)} + \frac{\cos(\pi x)}{s(s^2 + \pi^2)}$$

Conclusions:

Laplace transformation technique is a practical tool to solve P.D.E. with initial boundary value problems, and it might give strong motivation to consider solving all kind of P.D.E. with boundary value condition like heat equation in one or two dimension.

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