# COMPACTNESS OF FUZZY SETS

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**Abstract**

The objective of this paper is to study the compactness of fuzzy sets in fuzzy topological spaces, especially the connection between compact, closed and bounded fuzzy sets.

## 1-Introduction:

The concept of fuzzy sets was introduced initially by Zadeh in 1965. Since then, this concept is used in topology and some branches of analysis, many authors have expensively developed the theory of fuzzy sets and its application, [1].

Chang, C. L. in 1968 used the fuzzy set theory for defining and introducing fuzzy topological spaces, while Wong, C. K. in 1973, discussed the covering properties of fuzzy topological spaces, [2].

Ercey, M. A. in 1979, studied fuzzy metric spaces and its connection with statistical metric spaces, Ming P. P. and Ming L. Y. in 1980, used fuzzy topology to define neighborhood structure of fuzzy point and Moore-Smith convergence, Zike Deng in 1982, studied the fuzzy point and discussed the fuzzy metric spaces with the metric defined between two fuzzy points,[ 3].

The main objective of this paper is to study the relationship between closed and bounded fuzzy sets and the compactness of such fuzzy sets.

## 2- Basic Concepts in Fuzzy Topology [3], [4], [5]:

Chang C. L. in 1968 introduced the notion of fuzzy topological spaces, which is anon-empty set X together with a family of fuzzy sets in X which is closed under arbitrary union and finite intersection, as it is given precisely in the next definition.

We start first with the obvious definition of fuzzy topological spaces.

### *Definition (3.1),[3][7]:*

A family $\tilde{T}$ of fuzzy sets of X is called a fuzzy topology for X if and only if it satisfying the following conditions:

(a) $\varnothing, 1_X \in \tilde{T}$.where $\varnothing$ is the empty fuzzy set with membership function 0 and 1x is the universal set with membership function 1.

(b) If $\tilde{A}, \tilde{B} \in \tilde{T}$, then $\tilde{A} \cap \tilde{B} \in \tilde{T}$.

(c) If $\tilde{A}_i \in \tilde{T}, \forall\ i \in J$, where J is any index set, then $\bigcup_{i \in I} \tilde{A}_i \in \tilde{T}$.

$\tilde{T}$ is called fuzzy topology for X, and the pair $(X, \tilde{T})$ is a fuzzy topological space.

### Definition (3.2),[3]:

A fuzzy set $\tilde{A} \in X^*$ is said to be an open fuzzy set if $\tilde{A} \in \tilde{T}$ and is said to be closed fuzzy set if $\tilde{A}^c \in \tilde{T}$. $X^*$ is the set of all closed and bounded fuzzy subsets of X

### Definition (3.3), [2][4]:

Let $(X, \tilde{T})$ be a fuzzy topological space. A family of fuzzy subsets $\tilde{A}$ of $I^X$ is said to be a cover of a fuzzy set $\tilde{B}$ in X if and only if $\tilde{B} = \cup \tilde{A}$. If each member of the cover is a member of $\tilde{T}$, then this cover is said to be an open cover of $\tilde{B}$. A subcover of $\tilde{B}$ is a subfamily of the cover which is also a cover.

### Definition (3.4),[5]:

A fuzzy topological space is compact if and only if each open cover of the space has a finite subcover.

### Definition (3.5),[5]:

A fuzzy point $x_r$ of fuzzy set $\tilde{A}$ in a fuzzy topological space $(X, \tilde{T})$ is said to be fuzzy θ-cluster point (fuzzy δ-cluster point) of a fuzzy set $\tilde{A}$ in X if and only if fuzzy closure (respectively, interior of the closure) of every open q-neighborhood of $x_r$ is q-coincident with $\tilde{A}$

## 3- Compactness Fuzzy Set:

The compactness set is one of the fundamental aspects in topological space , in general, and of fuzzy set in particular, therefore, several approaches are proposed to study this subject.

Hence, in this section, we will give one of such approaches as a theorem. Also, we will stand and present some of the basic ideas for the construction and the proof of the completeness of fuzzy set, where the following abbreviation is used, $X^*$ is the set of all closed and bounded fuzzy subsets of X.

### Theorem(4.1):

If a fuzzy set is nonempty and is bounded below, then an infimum exist.

### Proof:

Let $\tilde{A}$ be a nonempty fuzzy set which is bounded below. Denote by C1 the fuzzy the set of all real numbers which are lower bounds of $\tilde{A}$, and let C2 consist of all other real numbers.

We may then show that [C1, C2] is a Dedekind.

1- The nature of the definition of C2 is a assurance that each real number is in C1 or C2.

2- Since $\tilde{A}$ is bounded below, C1 is non empty and since $\tilde{A}$ is not empty, not all real numbers are in C1. Hence C2 is not empty.

3- Let c1 be any member of C1, and let C be any number such C < C1.it follows that C is less than a lower bound of $\tilde{A}$ and hence is also a lower bound of $\tilde{A}$ .therefore c is a member of C1. This implies that all members of C2 exceed all members of C1.

4- Suppose C2 to have a least member. Denote it by c. then c is not a lower bound of $\tilde{A}$ , and, as a consequence, a number x of $\tilde{A}$ exists such that x < C. Also, between x and C exists another real number y, x < y < C, since y is less than c, it is necessarily in C1.

Also, since y exceeds x; a member of $\tilde{A}$ , it is not a lower bound and so is not in C1.the contradiction implies that C2 does not possess a least member. ∎

## Theorem (4.2):

The infimum of a nonempty fuzzy set $\tilde{A}$ is either a member of $\tilde{A}$ or a cluster point of $\tilde{A}$ .

## Proof:

Let $\tilde{b}$ = inf $\tilde{A}$ , be a fuzzy set with membership function

$$\mu_{\tilde{b}}(x) = \inf \mu_{\tilde{A}}(x), \forall\, x \in X$$

It is the minimum fuzzy point of $\tilde{A}$ , if $\tilde{b}$ is a member of $\tilde{A}$

While if $\tilde{b}$ is not a member of $\tilde{A}$ , then corresponding to each positive number $\varepsilon > 0$, the deleted fuzzy neighborhood $\tilde{N}^*(\tilde{b},\varepsilon)$ contains a point a of $\tilde{A}$ .

Hence b is a cluster point of $\tilde{A}$ . ∎

## Theorem (4.3):

A nonempty closed fuzzy set $\tilde{A}$ , which is bounded below possesses a minimum point.

## Proof:

By theorem (4.1) the infimum of the fuzzy set $\tilde{A}$ exists a fuzzy point $\tilde{b}$ which is either a point of $\tilde{A}$ or a cluster point of $\tilde{A}$ .

In the latter case it also follows that $\tilde{b}$ is in $\tilde{A}$, since $\tilde{A}$ is closed.

Hence the infimum of $\tilde{A}$ belongs to $\tilde{A}$ and is, of course, the minimum member of $\tilde{A}$. ■

## Theorem (4.4):

Any family of disjoint fuzzy intervals is countable.

## Proof:

Let $\{\tilde{I}_n\}$ be a sequence of fuzzy intervals hence, for all $\alpha \in (0, 1]$, let $\tilde{I}_n$ = $[\underline{I_n}, \overline{I_n} I]$ and hence, their is an equivalent two sequence of nonfuzzy intervals $\{\underline{I_n}\}$ and $\{\overline{I_n}\}$ which are countable hence $\{\tilde{I}_n\}$ is countable. ■

Similarly, using the idea of $\alpha$-level sets, one can state and prove the following theorem:

## Theorem (4.5):

Any nonempty open fuzzy set is the union of a unique countable collection of disjoint open intervals.

## 4- The Main Results:

Denoting that member of the family which corresponds to the integer

n by $A_n$, we may represent the sequence by $A_1$, $A_2$, …, $A_n$,…; or by the symbol $\{A_n\}$,

If a sequence of fuzzy sets of points has the property that $A_{n+1} \subset A_n$ for each n, then the sequence is referred to as decreasing or nested, concerning a nested sequence of sets each of which is closed, we state the following.

## Theorem (5.1):

If $\{\tilde{F}_n\}$ is a nested sequence of non empty, closed, and bounded fuzzy set, then the fuzzy set $\tilde{F} = \bigcap\limits_{n=1}^{\infty} \tilde{F}_n$ is non empty, where $\mu_{\tilde{F}}(x) = \mu_{\bigcap\limits_{n=1}^{\infty} \tilde{F}_n}(x)$.

## Proof:

By theorem (4.3) each of the fuzzy set $\tilde{F}_n$ possesses a maximum point.

Let $\tilde{X}_n = \max \tilde{F}_n$, for each n

$\mu_{\tilde{X}_n}(x) = \max \mu_{\tilde{F}_n}(x)$,     from     the hypothesis

$\tilde{F}_{n+1} \subset \tilde{F}_n$, $\mu_{\tilde{F}_{n+1}}(x) < \mu_{\tilde{F}_n}(x)$. It follows that $\tilde{X}_{n+1} \subseteq \tilde{X}_n$, and for each positive

integer q it is the case that $\tilde{X}_n \tilde{\in} \tilde{F}_q$, for all $n \geq q$.

Denote the fuzzy set of numbers $\tilde{X}_n$ by T, because $T \subset \tilde{F}_l$ and $\tilde{F}_l$ is bounded, it follows that T is bounded and, particular, is bounded below. Denoting the infimum of T by c. We shall show that $c \in \tilde{F}_n$, for all n.

Let q denote an arbitrary positive integer, and consider the subset of T defined by $\tilde{T}_q = \{\tilde{X}_n : n \geq q\}$, since c is the infimum of T, it is also the infimum of $\tilde{T}_q$, by theorem (4.2) it follows that c $\in \tilde{T}_q$. Also, since:

$$\tilde{T}_q \subset \tilde{F}_q, \text{ we have } \mu_{\tilde{T}_q}(x) \leq$$

$\mu_{\tilde{F}_q}(x)$.   ■

Now, consider the following remarks:

### *Remarks (5.1):*

A. Let $\tilde{F}_n = \tilde{\varnothing}$, for all positive integers n where $\varnothing$ is the empty fuzzy set with membership function 0. Then $\tilde{F}_n$ is closed and bounded, and $\{\tilde{F}_n\}$ is nested. However, the fuzzy set $\tilde{F} =$

$$\bigcap_{n=1}^{\infty} \tilde{F}_n, \text{ with membership function}$$

$$\mu_{\tilde{F}}(x) = \mu_{\bigcap\limits_{n=1}^{\infty} \tilde{F}_n}(x)$$

is empty since the $\tilde{F}_n$ are not non empty.

B. Let $\tilde{F}_n = \{(x, \mu_{\tilde{F}}(x), \text{where } x \in X \geq n$ is the universal set with membership function 1.} here the sequence $\{\tilde{F}_n\}$ is nested and $\tilde{F}_n$ is nonempty and closed for each n. however, each $\tilde{F}_n$ is unbounded. It is easy to see that $\tilde{F}$

$$= \bigcap_{n=1}^{\infty} \tilde{F}_n, \text{ with membership function}$$

$$\mu_{\tilde{F}}(x) = \mu_{\bigcap\limits_{n=1}^{\infty} \tilde{F}_n}(x) \text{ is empty.}$$

C. Let $\tilde{F}_n = (0, 1/n]$, the sequence $\{\tilde{F}_n\}$ is obviously nested, each $\tilde{F}_n$ is nonempty and bounded but not closed. The set $\tilde{F} = \bigcap_{n=1}^{\infty} \tilde{F}_n$, with membership function $\mu_{\tilde{F}}(x) =$

$\mu_{\bigcap\limits_{n=1}^{\infty} \tilde{F}_n}(x)$, seems to be empty.

D. Let $\tilde{F}_n = [2n, 2n+1]$. Here each $\tilde{F}_n$ is nonempty closed, and bounded, but the sequence $\{\tilde{F}_n\}$ is not nested. The

set $\tilde{F} = \bigcap\limits_{n=1}^{\infty} \tilde{F}_n$, with membership

function $\mu_{\tilde{F}}(x) = \mu_{\bigcap\limits_{n=1}^{\infty} \tilde{F}_n}(x)$ is empty.

## Therem (5.2):

If $L^*$ is any open covering of a fuzzy set $\tilde{A}$, then there exists a countable subfamily of L which also covers the fuzzy set $\tilde{A}$.

## Proof:

Let a denote any member of the given fuzzy set $\tilde{A}$. Then a fuzzy set $\tilde{G}_a$ of L exists such that $a \in \tilde{G}_a$. Further, since $\tilde{G}_a$ is open, a neighborhood N(a, ε) exists such that $N(a, ε) \subset \tilde{G}_a$.

Now, let $r_1$ and $r_2$ designate two rational numbers with the property that a − ε < $r_1$ < a < $r_2$ < a+ ε.

It is then the case that the interval $I_a = (r_1, r_2)$ is such that $a \in I_a$ and $I_q \subset \tilde{G}_a$.

Hence, in this manner we may associate with each member a of the fuzzy set $\tilde{A}$ an open interval $I_a$ with rational end points.

Since the fuzzy set of all possible intervals with rational end points is

countable, if follows that the fuzzy set $\tilde{B} = \{I_a : a \in \tilde{A}\}$ is also countable.

Each interval $I_a$ is contained in at least one of the open fuzzy sets of L and denote one such by $G_{a'}$. In this way a subfamily $L'$ of L is constructed with the property that with each interval $I_a$ of the countable fuzzy set $\tilde{B}$ is associated exactly one member $G_{a'}$ of $L'$.

Consequently $L'$ is countable and, moreover, covers A since for each member a of $\tilde{A}$ we have $a \in I_a$ and $I_a \subset G_{a'}$. ∎

## Theorem (5.3):

Let X = lR be the universal set, then a fuzzy subset of X is closed and bounded then it is compact.

## Proof:

Let $\tilde{A} \in X$ be a closed and bounded fuzzy subset of X and a family of a fuzzy sets A is a cover of a fuzzy set $\tilde{A}$ if and only if $\tilde{B} \subseteq \cup\{\tilde{A} \mid \tilde{A} \in A\}$

It is an open cover if and only if each member of A is an open fuzzy set $\tilde{A}$.

Because of the (Lindelof theorem in fuzzy sets), we may assume, without

loss of generality, that $\tilde{B}$ be a countable and thereby denote its members by $\tilde{G}_1$, $\tilde{G}_2$, ..., $\tilde{G}_n$, ... and define the fuzzy set:

$$\tilde{K}_n = \bigcup_{i=1}^{n} \tilde{G}_i \quad \text{and} \quad \tilde{L}_n = \tilde{A} \cap y \tilde{K}_n$$

which is s also a fuzzy set of X with membership function, for any index J

$$\mu_{\tilde{L}_n} \tilde{G}_j(x) = \sup_{i \in J} \mu_{\tilde{G}_i}(x), x \in X$$

$$\mu_{\tilde{L}_n}(x) = \text{Min} \{ \mu_{\tilde{A}}(x), y\mu_{\tilde{K}_n}(x) \}, \text{ for } n = 1, 2, \dots$$

and observe that, because of the theorems on unions and intersection of closed and open fuzzy set, the fuzzy set $\tilde{K}_n$ are open and the fuzzy set $\tilde{L}_n$ closed for all values of n. further, it is the case that:

$$\tilde{K}_n \subseteq \tilde{K}_{n+1}, \text{ then } \mu_{\tilde{K}_n}(x) \leq \mu_{\tilde{K}_{n+1}}(x), \forall x \in X$$

and from this follows

$$\tilde{L}_{n+1} \subseteq \tilde{L}_n, \text{ then } \mu_{\tilde{L}_{n+1}}(x) \leq \mu_{\tilde{L}_n}(x), \forall x \in X, \text{ for all n.}$$

Assume now that none of the fuzzy sets $\tilde{L}_n$ is empty fuzzy set, then:

$$\mu_{\tilde{L}_n}(x) = 1, \text{ since } \tilde{L}_n \subset \tilde{A}$$

and hence:

$$\mu_{\tilde{L}_n}(x) < \mu_{\tilde{A}}(x), \ x \in X$$

$1 < \mu_{\tilde{A}}(x)$, and hence $\tilde{A}$ is bounded

It follows then that the sequence $\{ \tilde{L}_n \} \subset \{ \tilde{L}_{n+1} \}$, n = 1, 2, ...; i.e.,

$$\mu_{\tilde{L}_n}(x) < \mu_{\tilde{L}_{n+1}}(x), \ x \in X, \text{ for all n.}$$

Satisfies the hypotheses of the theorem (5.2)

$$\tilde{L}_{n+1} = \bigcap_{n=1}^{\infty} \tilde{L}_n$$

$$\mu_{\tilde{L}_{n+1}}(x) = \mu_{\bigcap_{n=1}^{\infty} \tilde{L}_n}(x)$$

$$= \inf_{n} \mu_{\tilde{L}_n}(x), \forall x \in X$$

Therefore, for some positive integer q the fuzzy set $\tilde{L}_q = \tilde{A} \cap \xi \tilde{K}_q$ is empty.

$$\mu_{\tilde{L}_q}(x) = \min\{ \mu_{\tilde{A}}(x), \xi\mu_{\tilde{K}_q}(x) \}$$

$= 0$, for any index j, $\forall$ j = 1, 2, ..., q

Hence, $\tilde{A} \subset \tilde{K}_q = \bigcup_{i=1}^{q} \tilde{G}_i$, i.e.,

$$\mu_{\tilde{A}}(x) < \mu_{\tilde{K}_q}(x) = \mu_{\bigcup_{i=1}^{q} \tilde{G}_i}(x)$$

$$= \sup_{i \in j} \mu_{\tilde{G}_i}(x), \; \forall$$

$x \in X$

$$\therefore \tilde{A} \subset \tilde{C} \text{ (since } \mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$$

$$)$$

$\tilde{A}$ may be covered by the set $\tilde{C}$.

Let $\tilde{B}$ denote the family of open fuzzy sets of fuzzy points $\tilde{G}_n$, defined by:

$$\tilde{G}_n = \{(x, \mu_{\tilde{G}_n}(x)) : x \in X, -n \leq$$

$\mu_{\tilde{G}_n}(x) \leq n$, for each positive integer n$\}$

It is obvious that $\tilde{B}$ is an open covering of the fuzzy set of all real numbers and hence of any fuzzy set of real numbers suppose now that $\tilde{A} \in X^*$ is some compact fuzzy set, $\tilde{A} \in \tilde{T}$.

Then, since any open covering of $\tilde{A}$ possesses a finite sub family which also covers $\tilde{A}$. It follows, in particular, that this is true of $\tilde{B}$.

Consequently, a finite collection of intervals $\tilde{G}_n$ covers the fuzzy set $\tilde{A}$, and, if denotes the maximum subscript for this finite family, then clearly the open interval:

$$\tilde{G}_{n_0} = \{(x, \mu_{\tilde{G}_{n_0}}(x)) : x \in X;$$

$$-n_0 \leq \mu_{\tilde{G}_{n_0}}(x) \leq n_0\}$$

covers $\tilde{A}$.

This implies that a compact fuzzy set is bounded.

To show that $\tilde{A}$ is necessarily closed, let c be a real number and consider the family of closed fuzzy sets:

$$\tilde{F}_n = \{(x, \mu_{\tilde{F}_n}(x) \mid x \in X, c - 1/n$$

$$\leq \mu_{\tilde{F}_n}(x) \leq c + 1/n\}, n = 1, 2, \ldots$$

The fuzzy sets $\tilde{H}_n = \xi \tilde{F}_n, \; \forall n = 1, 2, \ldots$

$$\mu_{\tilde{H}_n}(x) = \xi \mu_{\tilde{F}_n}(x), \; \forall \; n = 1, 2,$$

$\ldots$

Then constitute a family of open fuzzy sets. It is obvious that the set $\bigcap_{n=1}^{\infty} \tilde{F}_n$

$$\mu_{\bigcap_{n=1}^{\infty} \tilde{F}_n}(x) = \inf_{i \in j} \mu_{\tilde{F}_i}(x), x \in X$$

for j any index set consists of the single points, and, since c in net in $\tilde{A}$, it follows that:

$$\tilde{A} \subset \xi \bigcap_{n=1}^{\infty} \tilde{F}_n$$

i.e.,

$$\mu_{\tilde{A}}(x) < \xi \mu_{\bigcap_{n=1}^{\infty} \tilde{F}_n}(x)$$

$$= \xi \inf_{i \in j} \mu_{\tilde{F}_i}(x), \quad \forall \ x \ \in$$

X…(1)

where:

$$\mu_{\tilde{A}}(x) < \mu_{\bigcup_{i=1}^{\infty} \tilde{H}_i}(x)$$

$$= \sup_{i \in j} \mu_{\tilde{H}_i}(x), \quad \forall \ x \ \in$$

X…..(2)

Thus the fuzzy set $\tilde{A}$ is covered by the family $\Delta$.

The compactness of $\tilde{A}$ implies that a finite subfamily of $\Delta$ exists which also covers $\tilde{A}$.

Therefore a positive integer $n_1$ exists such that each point of $\tilde{A}$ is contained in at least one of the open fuzzy sets $\tilde{H}_1$, $\tilde{H}_2$, …, $\tilde{H}_n$, …; then, no point of $\tilde{A}$ is contained in:

$$\tilde{F}_{n_1} = \{(x, \ \mu_{\tilde{F}_{n_1}}(x)) : x \in X, c - 1/n_1 \le \mu_{\tilde{F}_{n_1}}(x) \le c + 1/n_1\}$$

and from this it follows that the point c is not a cluster point of the fuzzy set $\tilde{A}$.

Thus it is proved that any point which is not a point of the fuzzy set $\tilde{A}$ is also not a cluster point of $\tilde{A}$.

All cluster points of $\tilde{A}$ are, therefore, points of $\tilde{A}$ itself.

Hence $\tilde{A}$ is closed.

## تراص المجموعات الضبابية

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