A Proposed Technique for Solving Linear Fractional Programming

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Abstract

Linear fractional programming (LFP) problems are useful tools in production planning, financial and corporate planning, health care and hospital planning and as such have attracted considerable research interest. The paper presents a new approach for solving a fractional linear programming problem in which the objective function is a linear fractional function, while the constraint functions are in the form of linear inequalities. We illustrate a number of numerical examples to demonstrate a proposed technique. We then compared proposed technique with Cooper's method in the literature for solving (LFP) problems

Keyword: ***Optimal Solution, Linear Fractional Programming ,Charnes& Cooper Transformation, New Approach***

طريقة مقترحة لحل أنموذج البرمجة الخطية بشكله الكسري

الخلاصة

إن الأسلوب الكسري للأنموذج الرياضي يعتمد على الأدوات والأساليب الحديثة المهمة في عملية تخطيط الإنتاج ، التخطيط المالي والاقتصادي والرعاية الصحية في المستشفيات، وعلى هذا النحو فقد جذبت اهتماماً وقدراً كبيراً في عملية البحث والتطوير.ينفرد البحث بنهج جديد أو تقنيه جديدة لحل مشكلة البرمجة الخطية الكسرية التي تكون دالة الهدف فيها دالة كسرية ، وأما القيود فهي خطية وعلى شكل متراجحات ، ولتوضيح آلية العمل سنقوم بإعطاء أمثلة رقمية لإثبات جودة ودقة التقنية الجديدة مع الأخذ بنظر الاعتبار مقارنتها مع طريقة كوبر من اجل الحصول على الحل الأمثل لمشكلة البرمجة الخطية الكسرية

الكلمات المفتاحية: الحل الأمثل ، البرمجة الخطية الكسرية ، تحويل كوبر ، التقنية الجديدة

1. INTRODUCTION :

In this section, we establish the relationship between LP and LFP problems, and we consider the following (LFP).

$$(LFP) \qquad Maximize \quad F(x) = \frac{c^T x + \alpha}{d^T x + \beta} \quad \text{.....................}(1)$$

Subject to

$$x \in X = \{x \in R^n : Ax \le b,\ x \ge 0\} \text{....................}(2)$$

Where

*A : is a (m*n) Matrix ; c,d,x are(n*1) vectors*

*b :is an (m*1) vector and* α, β *are scalars*

It is assumed that the feasible regions nonempty and bounded and the denominator

is satisfy $\frac{\beta(Ax - b)}{\beta(d^T x + \beta)} \le 0$ *dx+β>0. if dx+β≤0 then the condition*

2. Solving Approach :

Approach one: Charnes& Cooper Transformation to linear program ,we assume

including two Transformation $t = \frac{1}{d^T x + \beta}, Y = \frac{1}{d^T x + \beta} x, Y = tx \text{ and } x = \frac{Y}{t}$

(i) :Transformation of Objective Function (Z) then

$$F(x) = \frac{c^T x + \alpha}{d^T x + \beta} = t(c^T x + \alpha)$$

$$\because x = \frac{Y}{t} \quad then$$

$$F(Y) = t(c^T \frac{Y}{t} + \alpha) = c^T Y + \alpha t \,.........(3)$$

(ii) :Transformation of Constraints (S.T.)

$$\because Ax \leq b \rightarrow x = \frac{Y}{t} \quad then$$

$$A\frac{Y}{t} \leq b \quad then$$

$$AY \leq bt$$

$$AY - bt \leq 0................(4)$$

$$and$$

$$t = \frac{1}{d^T x + \beta} \rightarrow t(d^T x + \beta) = 1$$

$$\therefore t(d^T \frac{Y}{t} + \beta) = 1 \;\; then$$

$$d^T Y + \beta t = 1 \,..................(5)$$

From above equations(3),(4)and(5).we obtain the new(LP) model from (LFP)model as follow

$$\begin{aligned} &Max \quad F(Y,t) = c^T Y + \alpha t \\ &S.T. \\ &\qquad AY - bt \leq 0..........\mu \\ &\qquad d^T Y + \beta t = 1..........\lambda \\ &\qquad Y, t \geq 0 \end{aligned}$$

And the dual program is

$$Max \quad F(\mu,\lambda) = \lambda$$
$$S.T.$$
$$A\mu + d^T \lambda \geq c^T$$
$$-bY + \beta\lambda = \alpha$$
$$\mu \geq 0 , \lambda \text{ is } U.R.S.$$

***Approach two**: New technique transformation including two transformation*

(i) :Transformation of Objective Function (Z)

Multiplying both the denominator and numerator of equation(1) by β, we get

$$F(x) = \frac{\beta(c^T x + \alpha)}{\beta(d^T x + \beta)} \quad \text{adding \& substruuct } (d^T \alpha x) \text{ for the numerator}$$
$$F(x) = \frac{\beta c^T x + d^T \alpha x - d^T \alpha x + \beta\alpha}{\beta(d^T x + \beta)}$$
$$F(x) = \frac{(c^T \beta - d^T \alpha)x + (d^T x + \beta)\alpha}{\beta(d^T x + \beta)}$$
$$F(x) = \left(c^T - \frac{\alpha}{\beta} d^T\right)\frac{x}{(d^T x + \beta)} + \frac{\alpha}{\beta} \ldots\ldots\ldots\ldots\ldots\ldots(6)$$
$$Put \quad Y = \frac{x}{(d^T x + \beta)} , V = \left(c^T - \frac{\alpha}{\beta} d^T\right) \quad and \quad P = \frac{\alpha}{\beta}$$
$$F(Y) = VY + P \ldots\ldots\ldots\ldots\ldots(7)$$

(ii) :Transformation of Constraints (S.T.)

Multiplying equation(2) by β, we get

$$\frac{\beta(Ax - b)}{\beta(d^T x + \beta)} \le 0$$

$$\frac{\beta Ax - b\beta}{\beta(d^T x + \beta)} \le 0 \qquad \textit{adding \& substruuct } (d^T bx) \textit{ for the numerator}$$

$$\frac{\beta Ax + d^T bx - d^T bx - b\beta}{\beta(d^T x + \beta)} \le 0$$

$$\frac{\beta\left(Ax + d^T \frac{b}{\beta}\right)x}{\beta(d^T x + \beta)} - \frac{b}{\beta}\left(\frac{d^T x + \beta}{d^T x + \beta}\right) \le 0$$

$$\left(Ax + d^T \frac{b}{\beta}\right)\frac{x}{d^T x + \beta} \le \frac{b}{\beta} \;\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$Put\ G = \left(Ax + d^T \frac{b}{\beta}\right),\ Y = \frac{x}{d^T x + \beta},\ H = \frac{b}{\beta}$$

$$GY \le H \ldots\ldots\ldots\ldots(9)$$

From above equations(7)and(9).we obtain the new(LP) model from (LFP)model as follow

$$F(Y) = VY + P$$
$$S.T.$$
$$GY \le H$$
$$Y \ge 0$$

3. Numerical Examples

In this section, we will illustrate some numerical examples to demonstrate approaches.

Example (1) : (Charnes&*Cooper approach)*

$$Max\quad Z = \frac{x_1 + 2x_2}{2x_1 - x_2 + 2}$$
$$S.T. \quad -x_1 + 2x_2 \le 2$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

$$Let \quad t=\frac{1}{2x_1-x_2+2} \quad , \; Y_1=tx_1 \rightarrow x_1=\frac{Y_1}{t}$$
$$Y_2=tx_2 \rightarrow x_2=\frac{Y_2}{t}$$

Then the new(Z) is

$$F(Y,t)=t\left(\frac{Y_1}{t}+2\frac{Y_2}{t}\right)$$

$$F(Y,t)=Y_1+2Y_2$$

And the new constraints are

$$S.T$$
$$\rightarrow \left(-\frac{Y_1}{t}+2\frac{Y_2}{t}\right)\leq 2$$
$$\rightarrow \quad -Y_1+2Y_2\leq 2t$$

$$-Y_1+2Y_2-2t\leq 0$$

$$\rightarrow \left(\frac{Y_1}{t}+\frac{Y_2}{t}\right)\leq 4$$
$$\rightarrow \quad Y_1+Y_2\leq 4t$$

$$Y_1+Y_2-4t\leq 0$$

Also

$$\rightarrow \left(2\frac{Y_1}{t}-\frac{Y_2}{t}+2\right)=1$$

$$2Y_1 - Y_2 + 2t = 1$$

So, the new (LP) model is

$$\begin{aligned} & F(Y,t) = Y_1 + 2Y_2 \\ S.T. \quad & -Y_1 + 2Y_2 - 2t \leq 0 \\ & Y_1 + Y_2 - 4t \leq 0 \\ & 2Y_1 - Y_2 + 2t = 1 \\ & Y, t \geq 0 \end{aligned}$$

The initial basic feasible solution is given bellow

Tableau(1)

B.V	X_B	1	2	0	0	0	-M
		Y_1	Y_2	t	S_1	S_2	R_1
S_1	0	-1	2	-2	1		
S_2	0	1	1	-4		1	
R_1	1	2	-1	2			1
F(Y,t)	-M	1	2	0			

And the optimal solution is given bellow

Tableau(6)

B.V	X_B	1	2	0	0	0	-M
		Y_1	Y_2	t	S_1	S_2	R_1
Y_2	1	1	1		1		1
S_2	3	6			1	1	3
T	1	3/2		1	1/2		1
F(Y,t)	2	1			2		M+2

Now

$$x_1 = \frac{Y_1}{t} = \frac{0}{1} = 0$$
$$x_2 = \frac{Y_2}{t} = \frac{1}{1} = 1$$
$$Z(x) = \frac{x_1 + 2x_2}{2x_1 - x_2 + 2} = \frac{(0) + 2(1)}{2(0) - (1) + 2} = 2$$

Example (2) :(Charnes&Cooper approach)

$$\begin{aligned} &Max \quad Z = \frac{5x_1 + 6x_2}{2x_2 + 7} \\ &S.T. \qquad 2x_1 + 3x_2 \le 6 \\ &\qquad\qquad 2x_1 + x_2 \le 3 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

$$Let \quad t = \frac{1}{2x_2 + 7} \quad , \; Y_1 = tx_1 \rightarrow x_1 = \frac{Y_1}{t}$$
$$Y_2 = tx_2 \rightarrow x_2 = \frac{Y_2}{t}$$

Then the new(Z) is

$$F(Y,t) = t\left(5\frac{Y_1}{t} + 6\frac{Y_2}{t}\right)$$

$$F(Y,t) = 5Y_1 + 6Y_2$$

And the new constraints are

S.T

$$\rightarrow \left(2\frac{Y_1}{t}+3\frac{Y_2}{t}\right)\le 6$$

$$\rightarrow \quad 2Y_1+3Y_2\le 6t$$

$$2Y_1+3Y_2-6t\le 0$$

$$\rightarrow \left(2\frac{Y_1}{t}+\frac{Y_2}{t}\right)\le 3$$

$$\rightarrow \quad 2Y_1+Y_2\le 3t$$

$$2Y_1+Y_2-3t\le 0$$

Also

$$\rightarrow \left(2\frac{Y_2}{t}+7\right)=1$$

$$2Y_2+7t=1$$

So, the new (LP) model is

$$\begin{aligned} F(Y,t) &= 5Y_1+6Y_2 \\ \text{S.T.} \quad 2Y_1+3Y_2-6t &\le 0 \\ 2Y_1+Y_2-3t &\le 0 \\ 2Y_2+7t &= 1 \\ Y,t &\ge 0 \end{aligned}$$

The initial basic feasible solution is given bellow

Tableau(1)

B.V	X_B	5	6	0	0	0	-M
		Y_1	Y_2	t	S_1	S_2	R_1
S_1	0	2	3	-6	1		
S_2	0	2	1	-3		1	
R_1	1	0	2	7			1
F(Y,t)	-M	-5	-6	0			

And the optimal solution is given bellow

Tableau(5)

B.V	X_B	5	6	0	0	0	-M
		Y_1	Y_2	t	S_1	S_2	R_1
Y_2	0.15		1		0.35	-0.35	0.15
Y_1	0.075	1			-0.325	0.825	0.075
T	0.10			1	-0.1	0.1	0.1
F(Y,t)	1.275				0.475	2.025	1.275+M

Now

$$x_1 = \frac{Y_1}{t} = \frac{0.075}{0.10} = 0.75 = \frac{3}{4}$$

$$x_2 = \frac{Y_2}{t} = \frac{0.15}{0.10} = 1.5 = \frac{3}{2}$$

$$Z(x) = \frac{5x_1 + 6x_2}{2x_1 + 7} = \frac{5\left(\frac{3}{4}\right) + 6\left(\frac{3}{2}\right)}{2\left(\frac{3}{2}\right) + 7} = \frac{51}{40}$$

Resolve Example (1) by using (New technique 1)

$$Max \quad Z = \frac{x_1 + 2x_2}{2x_1 - x_2 + 2}$$
$$S.T. \qquad -x_1 + 2x_2 \le 2$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

We have

$$c^T = (1,2)\ ,\ d^T = (2,-1)\ ,\alpha = 0\ ,\beta = 2$$
$$A_1 = (-1,2)\ ,b_1 = 2$$
$$A_1 = (1,1)\quad ,b_1 = 4$$

Where

A_1& b_1 is related to the first constraint

A_2& b_2 is related to the second constraint

So ,we have the objective function from equation (7)

$$F(Y) = \left[(1,2) - \frac{o}{2}(2,-1)\right]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \frac{o}{2}$$

$$F(Y) = Y_1 + 2Y$$

Now the first constraint ,we get from equation(9)

$$\left[(-1,2) + \frac{2}{2}(2,-1)\right]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \le \frac{2}{2}$$

$$Y_1 + Y_2 \leq 1$$

Similarly second constraint ,we get

$$\left[(1,1) + \frac{4}{2}(2,-1)\right]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \leq \frac{4}{2}$$

$$5Y_1 - Y_2 \leq 2$$

So, the new (LP) model is

$$\begin{aligned} F(Y) &= Y_1 + 2Y_2 \\ S.T. \quad Y_1 + Y_2 &\leq 1 \\ 5Y_1 - Y_2 &\leq 2 \\ Y_1, Y_2 &\geq 0 \end{aligned}$$

The tableau(2) is represent the optimal solution (final tableau)is given below

Tableau(2)

B.V	X_B	1	2	0	0
		Y_1	Y_2	S_1	S_2
Y_2	1	1	1	1	
S_2	4	7		2	1
F(Y)	2	1		2	

Now

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{\beta \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}{1 - d^T \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{1 - (2,-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}{2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Z(x) = \frac{x_1 + 2x_2}{2x_1 - x_2 + 2} = \frac{(0) + 2(1)}{2(0) - (1) + 2} = 2$$

Resolve Example (2) by using (New technique 1)

$$\begin{aligned} &Max \quad Z = \frac{5x_1 + 6x_2}{2x_2 + 7} \\ &S.T. \qquad 2x_1 + 3x_2 \le 6 \\ &\qquad\qquad 2x_1 + \ x_2 \le 3 \\ &\qquad\qquad\quad x_1, x_2 \ge 0 \end{aligned}$$

We have

$$\begin{aligned} &c^T = (5,6)\ ,\ d^T = (0,2)\ \ ,\alpha = 0\ \ ,\beta = 7 \\ &A_1 = (2,3)\ \ ,\ b_1 = 6 \\ &A_1 = (2,1)\ \ ,\ b_1 = 3 \end{aligned}$$

Where

A_1& b_1 is related to the first constraint

A_2& b_2 is related to the second constraint

So ,we have the objective function from equation (7)

$$F(Y) = \left[(5,6) - \frac{0}{1}(0,2) \right] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \frac{0}{1}$$

$$F(Y) = 5Y_1 + 6Y$$

Now the first constraint ,we get from equation(9)

$$\left[(2,3) + \frac{6}{7}(0,2) \right] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \leq \frac{6}{7}$$

$$14Y_1 + 33Y_2 \leq 6$$

Similarly second constraint ,we get

$$\left[(2,1) + \frac{3}{7}(0,2) \right] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \leq \frac{3}{7}$$

$$14Y_1 + 13Y_2 \leq 3$$

So, the new (LP) model is

$$\begin{aligned} & F(Y) = 5Y_1 + 6Y_2 \\ S.T. \quad & 14Y_1 + 33Y_2 \leq 6 \\ & 14Y_1 + 13Y_2 \leq 3 \\ & Y_1, Y_2 \geq 0 \end{aligned}$$

The tableau(2) is represent the optimal solution (final tableau)is given below

Tableau(2)

B.V	X_B	5	6	0	0
		Y_1	Y_2	S_1	S_2
Y_2	0.15		1	0.05	-0.05
Y_1	0.075	1		-0.046	0.11
F(Y)	1.275			0.067	0.283

Now

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{\beta \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}{1 - d^T \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{7 \begin{bmatrix} 0.15 \\ 0.075 \end{bmatrix}}{1 - (0,2) \begin{bmatrix} 0.15 \\ 0.075 \end{bmatrix}} = \begin{bmatrix} 0.75 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 3/2 \end{bmatrix}$$

$$Z(x) = \frac{5x_1 + 6x_2}{2x_2 + 7} = \frac{5(3/4) + 6(3/2)}{2(3/2) + 7} = \frac{51}{40}$$

4. COMPARISON

In this section, we give a comparison chart to show the efficiency of new technique and (QM-Software)with the Cooper's procedure. To find the duration of implementation code we use "Run Time" command. We use the following computer configuration. Processor: x86 Family 6 Model 15 Stepping 13 GenuineIntel 2.00GHZ, Memory(RAM):2.00 GB, System type: X86-based PC

Numerical Examples	Methods	Iteration use	Computer Time taken
1	Cooper's procedure	Six	0.59 sec.
	new technique	two	0.11 sec.
2	Cooper's procedure	Five	0.50 sec.
	new technique	two	0.11 sec.

5.Conclusion

Our aim was to develop an easy technique for solving LFP problems. In this study, we have introduced new technique, which converts the LFP problem into a single LP problem. A method for solving linear fractional functions with constraint functions in the form of linear inequalities is given. The proposed method differs from the earlier methods as it is based upon solving the problem algebraically using the concept new transformation. The method appears simple to solve any linear fractional programming problem of any size. We also compared these results obtained by proposed method with cooper's method.

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