Variational Iterational Method for Solving Some Real Life Applications

Dunya Mohee Haider
dunyamoheehaydee@gmail.com
Madent Alelm College Unversity
Baghdad -Iraq

Ali Abdul Kadhum Ruhaima
ali.abd.eng@nuc.edu.iq
Al Nisour College University
Baghdad -Iraq

Abstract

In this work, Variational Iterational Method ,which is a modified Lagrange Multiplier, was used to solve a nonlocal problems arising in thermoelastisity , where the one dimensional nonhomogeneous Heat equation was introduced together with the initial condition and the homogenous nonlocal conditions to reach the analytical solution.

Keywords: Variational iteration method, Lagrange multiplier, nonlocal condition.

طريقة التكرار التغايري لحل بعض التطبيقات الحياتية

علي عبد الكاظم رحيمة
كلية النسور الجامعة

دنيا محي حيدر
كلية مدينة العلم الجامعة

الخلاصة:

تم في هذا العمل ، استخدام طريقة التكرار التغايري، والتي هي عبارة عن مضروب لاكرانج المطور، لحل المسائل اللا محلية الشروط التي تظهر في مشاكل المرونة الحرارية، حيث تم تقديم معادلة الحرارة اللا متناجسة ذات البعد الواحد مع الشرط الابتدائي والشروط اللا محلية المتجانسة وصولا الى الحل التحليلي.

الكلمات المفتاحية : طريقة التكرار التغايري ، مضروب لاكرانج ، الشرط اللا محلي

Introduction

The variational iteration method, which is a modified general Lagrange multiplier has been shown to solve effectively, easily, and accurately a large class of linear and nonlinear problems with approximation converging rapidly to accurate solutions [4], [7] recently introduced variational iteration method which gives rapidly convergent successive approximations of the exact solution if such a solution exists. This method has proved successful in deriving analytical solutions of linear and nonlinear differential equations. In their paper, Jafari, Hossinzadeh and Salehpoor solved Gas Dynamics Equation using variational iteration method, [4]. Variational iteration method was used to solve some types of Volterra's integro-differential equations[1]. In recent years, problems with integral conditions have received an increasing attention. The physical significance of integral conditions (mean, total flux, total energy, total mass, moments,…) has served as a fundamental reason for the interest carried to this type of problem [4].

1. The Variational Iteration Method [6]

To illustrate the basic idea of this technique, we consider the following general nonlinear equation:

$$L[u(x,t)] + N[u(x,t)] = g(x,t) \quad \ldots (1)$$

Where L is a linear operator, N is a nonlinear operator, g is a given function of x and t and u is the unknown function that must be determined for $t \geq t_0$.

The basic character of the variational iteration method is to construct a correction function for equation (1) which reads

$$u_{i+1}(x,t) = u_i(x,t) + \int_{t_0}^{t} \lambda[Lu_i(x,s) + N\tilde{u}_i(x,s) - g(x,s)]ds, \quad \ldots (2)$$

Where λ is a general Lagrange multiplier which can be identified optimally via variational theory, u_i is the i th approximate solution, and $\tilde{u}_i$ denotes a restricted variation, i.e., $\delta\tilde{u}_i = 0$, [6]. Then we substitute λ into the following iteration formula:

$$u_{i+1}(x,t) = u_i(x,t) + \int_{0}^{t} \lambda[Lu_i(x,s) + Nu_i(x,s) - g(x,s)]ds, \qquad i = 0,1,\cdots \quad \ldots (3)$$

Where u_0 is the initial approximation to the solution of equation (1)

2. The Variational Iteration Method for Solving Heat Equation with Homogeneous Nonlocal Conditions [6]

In this section, is used the variatianal iteration method for solving the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \leq x \leq \ell, \qquad 0 < t \leq T \quad \ldots (4)$$

together with the initial condition

$$u(x,0) = r(x), \qquad 0 \leq x \leq \ell \quad \ldots (5)$$

the homogeneous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \leq t \leq T \quad \ldots (6)$$

and the homogeneous nonlocal condition

$$\int_0^{\ell} u(x,t)dx = 0, \qquad 0 \leq t \leq T \quad \ldots (7)$$

where γ is a nonzero constant, f is a known function of x and t, and r is a given function of x that must satisfy the following compatibility conditions

$$r'(0) = \int_0^{\ell} r(x)dx = 0$$

In order to use the variational iteration method to solve such type of nonlocal problems one must rewrite equation (4) as

$$L(u(x,t)) + N(u(x,t)) = f(x,t)$$

where $L = \frac{\partial}{\partial t}$ and $N = -\gamma^2 \frac{\partial^2}{\partial x^2}$.

Therefore equation (2) becomes:

$$u_{i+1}(x,t) = u_i(x,t) + \int_0^t \lambda(s,t)\left[\frac{\partial u_i(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds \qquad \dots (8)$$

Where λ is the Lagrange multiplier. Thus by taking the variation of the above equation one can have:

$$\delta u_{i+1}(x,t) = \delta u_i(x,t) + \delta\int_0^t \lambda(s,t)\left[\frac{\partial u_i(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds$$

Then by using the integration by parts one can obtain

$$\delta u_{i+1}(x,t) = \delta u_i(x,t) + \lambda(s)\delta u_i(x,s)\big|_{s=t} - \delta\int_0^t \lambda'(s)u_i(x,s)ds + \delta\int_0^t\left[-\gamma^2\lambda(s)\frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - \lambda(s)f(x,s)\right]ds$$

$$= \delta u_i(x,t)[1+\lambda(s)]_{s=t} - \int_0^t \lambda'(s)\delta u_i(x,s)ds + \delta\int_0^t\left[-\gamma^2\lambda(s)\frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - \lambda(s)f(x,s)\right]ds$$

The stationary conditions will be:

$$[1+\lambda(s)]_{s=t} = 0 \qquad \dots (9)$$

and

$$\lambda'(s) = 0, \qquad 0 \le s \le t$$

The solution of the above differential equation is

$$\lambda(t) = A$$

where A is an arbitrary constant. To find the value of A, substitute λ into equation (9) to get:

$$1 + A\big|_{s=t} = 0$$

Therefore

$$\lambda(s) = A = -1$$

By substituting $\lambda = -1$ into equation (3) one can obtain the following iteration formula:

$$u_{i+1}(x,t) = u_i(x,t) - \int_0^t \left[\frac{\partial u_i(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 u_i(x,s)}{\partial x^2} - f(x,s) \right] ds \qquad \dots (10)$$

For simplicity, let $u_0(x,t) = r(x)$, then

$$u_0(x,0) = r(x), \qquad 0 \le x \le \ell$$

$$\left.\frac{\partial u_0(x,t)}{\partial x}\right|_{x=0} = r'(x)\big|_{x=0} = r'(0) = 0, \qquad 0 \le t \le T$$

and

$$\int_0^\ell u_0(x,t)dx = \int_0^\ell r(x)dx = 0, \qquad 0 \le t \le T$$

Therefore $u_0(x,t) = r(x)$ is the initial approximation of the solution of equation (4) that satisfies the initial condition, the Neumann condition and the nonlocal condition given by equation (5)-(7).

Then by setting $i = 0$ into equation (10) one can have:

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 u_0(x,s)}{\partial x^2} - f(x,s) \right] ds$$

$$= r(x) - \int_0^t \left[-\gamma^2 r''(x) - f(x,s) \right] ds$$

$$= r(x) + \gamma^2 r''(x)t + \int_0^t f(x,s)ds$$

By setting $i = 1$ in equation (10) and by substituting $u_1(x,t)$ in it, one can get $u_2(x,t)$. By continuing in this manner one can get:

$$u(x,t)=\lim_{i\to\infty}u_i(x,t)$$

is the solution of the nonlocal problem given by equations (4)-(7).

Next, to show the convergence of the variational iteration method for solving the nonlocal problem given by equations (4)-(7), we gives the following theorem. This theorem is a special case of theorem (1) that appeared in [2, pp17].

Theorem 1: [3]

Let $u\in C^2(\Omega)$ be the exact solution of the nonlocal problem given by equations (4)-(7) and $u_i\in C^2(\Omega)$, where $\Omega=\{(x,t)|0\le x\le \ell,0\le t\le T\}$, be the obtained solution of the sequence defined by equation (2.10) with $u_0(x,t)=r(x)$. If

$$E_i(x,t)=u_i(x,t)-u(x,t) \qquad i=0,1,\dots$$

and

$$\left\|\frac{\partial^2 E_i(x,t)}{\partial x^2}\right\|_2\le\|E_i(x,t)\|_2$$

Where $\|E_i(x,t)\|_2=\int_0^T\int_0^\ell|E_i(x,t)|^2\,dxdt$,

Then the sequence defined by equation (2.10) converges to u .

<u>Proof</u>

Since u is the exact solution of equation (2.4), then

$$u_{i+1}(x,t)-u(x,t)=u_i(x,t)-u(x,t)-\int_0^t\left[\frac{\partial u_i(x,s)}{\partial s}-\gamma^2\frac{\partial^2u_i(x,s)}{\partial x^2}+f(x,s)\right]ds$$
$$+\int_0^t\left[\frac{\partial u(x,s)}{\partial s}-\gamma^2\frac{\partial^2u(x,s)}{\partial x^2}-f(x,s)\right]ds$$
$$=u_i(x,t)-u(x,t)-\int_0^t\left[\frac{\partial}{\partial s}\{u_i(x,s)-u(x,s)\}-\gamma^2\frac{\partial^2}{\partial x^2}\{u_i(x,s)-u(x,s)\}\right]ds$$

But

$$E_i(x,0)=u_i(x,0)-u(x,0) \qquad i=0,1,\dots$$

Then

$$E_0(x,0) = u_0(x,0) - u(x,0) = r(x) - r(x) = 0$$

And from equation (2.10), one can have

$$u_{i+1}(x,0) = u_i(x,0), \qquad i = 0,1,\ldots$$

Therefore

$$u_i(x,0) = u_0(x,0), \qquad i = 0,1,\ldots.$$

Hence

$$E_i(x,0) = u_i(x,0) - u(x,0) = 0, \qquad i = 0,1,\ldots$$

And this implies that

$$E_{i+1}(x,t) = \gamma^2 \int_0^t \frac{\partial^2 E_i(x,s)}{\partial x^2} ds$$

Thus, according to norm properties, we have

$$\|E_{i+1}(x,t)\|_2 = \gamma^2 \left\| \int_0^t \frac{\partial^2 E_i(x,s)}{\partial x^2} ds \right\|_2 \le \gamma^2 \int_0^t \left\| \frac{\partial^2 E_i(x,s)}{\partial x^2} \right\|_2 ds$$

Hence

$$\|E_{i+1}(x,t)\|_2 \le \gamma^2 \int_0^t \|E_i(x,s)\|_2 ds$$

For $i = 0$ one can have:

$$\begin{aligned} \|E_1(x,t)\|_2 &\le \gamma^2 \int_0^t (\|E_0(x,s)\|_2) ds \\ &\le \gamma^2 \max_{(x,s)\in\Omega} \|E_0(x,s)\|_2 \int_0^t ds \\ &= \gamma^2 \max_{(x,s)\in\Omega} \|E_0(x,t)\|_2 t \end{aligned}$$

For $i = 1,$

$$\|E_2(x,t)\|_2 \le \gamma^2 \int_0^t (\|E_1(x,s)\|_2)ds$$

$$\le \gamma^4 \int_0^t (\max_{(x,s)\in\Omega} \|E_0(x,s)\|_2 s)ds$$

$$= \gamma^4 \max_{(x,s)\in\Omega} \|E_0(x,t)\|_2 \frac{t^2}{2!}$$

By continuing in this manner one can have:

$$\|E_i(x,t)\|_2 \le \gamma^{2i} \max_{(x,s)\in\Omega} \|E_0(x,t)\|_2 \frac{t^i}{i!}$$

By letting $i \to \infty$ one can obtain:

$$\|E_i(x,s)\|_2 \longrightarrow 0 \quad \text{as } i \longrightarrow \infty$$

And this implies that

$$E_i(x,s) \longrightarrow 0 \quad \text{as } i \longrightarrow \infty$$

Therefore

$$\lim_{i\to\infty} E_i(x,t) = 0$$

Which gives

$$\lim_{i\to\infty} u_i(x,t) = u(x,t)$$

3. The Mathematical Modeling of the thermoelectricity Problem [3]:

In this section we describe the mathematical modeling for a thermoelasticity rod problem. Let us consider a rod $0 \le x \le 1$, the temperature $v = v(x,t)$ and the transverse displacement $z = z(x,t)$. The thermoelasticity rod problem can be described by the coupled partial differential equations

$$\mu \frac{\partial^2 v(x,t)}{\partial x^2} = k \frac{\partial v(x,t)}{\partial t} + v_0 \beta \frac{\partial^3 z(x,t)}{\partial x^2 \partial t}, \qquad \dots (11)$$

$$\alpha \frac{\partial^4 z(x,t)}{\partial x^4} = \beta \frac{\partial^2 v(x,t)}{\partial x^2} \qquad \dots (12)$$

where μ is the thermal conductivity, k is the specific heat at constant strain, α is the flexural rigidity, β is a measure of the cross-coupling between thermal and mechanical efforts, v_0 is a uniform reference temperature.

If we suppose that the initial temperature of the rod is $r(x)$, and the initial displacement is $f(x)$; the ends $x=0$ and $x=1$ are clamped. Then

$$v(x,0) = r(x) \qquad \ldots (13)$$

$$z(x,0) = f(x) \qquad \ldots (14)$$

$$z(0,t) = \left.\frac{\partial z(x,t)}{\partial x}\right|_{x=0} = z(1,t) = \left.\frac{\partial z(x,t)}{\partial x}\right|_{x=1} = 0 \qquad \ldots (15)$$

Moreover if we assume that the average temperature in the rod $0 \le x \le 1$ is equal to $g_1(t)$. That is

$$\int_0^1 v(x,t)dx = g_1(t) \qquad \ldots (16)$$

and the difference between the heat exchange of the atmosphere on the end $x=0$ and the temperature on the end $x=1$ is equal to $g_2(t)$, then by using Newton's law one can have:

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} + v(0,t) - v(1,t) = g_2(t) \qquad \ldots (17)$$

We reformulate the problem given by equation (11)-(17) into an equivalent form where the coupled partial differential equations (11)-(12) is reduced to one partial differential equation. To do this we introduce a new unknown function u defined as follows:

$$u(x,t) = \frac{k}{v_0}\left[v(x,t) - v_0(x,t)\right] + \beta \frac{\partial^2 z(x,t)}{\partial x^2} \qquad \ldots (18)$$

where u is the entropy. Then

$$v_0 \frac{\partial u(x,t)}{\partial t} = k \frac{\partial v(x,t)}{\partial t} + v_0 \beta \frac{\partial^3 z(x,t)}{\partial x^2 \partial t} \qquad \ldots (19)$$

$$\mu \frac{\partial^2 v(x,t)}{\partial x^2} = v_0 \frac{\partial u(x,t)}{\partial t} \qquad \ldots (20)$$

By using equation (5), (11-12), one can get:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{k}{v_0}\frac{\partial^2 v(x,t)}{\partial x^2} + \beta \frac{\partial^4 z(x,t)}{\partial x^4}$$

$$= \frac{k}{v_0}\frac{v_0}{\mu}\frac{\partial u(x,t)}{\partial t} + \frac{\beta^2}{\alpha}\frac{\partial^2 v(x,t)}{\partial x^2}$$

$$=\frac{k}{\mu}\frac{\partial u(x,t)}{\partial t}+\frac{\beta^2}{\alpha}\frac{v_0}{\mu}\frac{\partial u(x,t)}{\partial t}$$

$$=\left[\frac{k}{\mu}+v_0\frac{\beta^2}{\alpha\mu}\right]\frac{\partial u(x,t)}{\partial t}$$

Therefore, the entropy u is a solution of the heat equation:

$$\mu\frac{\partial^2 u(x,t)}{\partial x^2}=\left[k+v_0\frac{\beta^2}{\alpha}\right]\frac{\partial u(x,t)}{\partial t} \quad \ldots (21)$$

To deduce the initial condition on the entropy u, we use the conditions given by equations (13)-(14) to get:

$$u_0(x)=\frac{k}{v_0}[r(x)-v_0]+\beta f''(x) \quad \ldots (22)$$

Then

$$u(x,0)=u_0(x) \quad \ldots (23)$$

To deduce the first boundary condition on the entropy u, we integrate u with respect to x from $x=0$ to $x=1$ to get:

$$\int_0^1 u(x,t)dx=\frac{k}{v_0}\left[\int_0^1 v(x,t)dx-v_0\right]+\beta\left[\left.\frac{\partial z(x,t)}{\partial x}\right|_{x=1}-\left.\frac{\partial z(x,t)}{\partial x}\right|_{x=0}\right]$$

By using equation (15)-(16) one can have:

$$\int_0^1 u(x,t)dx=\frac{k}{v_0}[g_1(t)-v_0] \quad \ldots (24)$$

Let $\theta_1(t)=\frac{k}{v_0}[g_1(t)-v_0]$

$$\int_0^1 u(x,t)dx=\theta_1(t) \quad \ldots (25)$$

which is the average entropy. To conclude the second boundary condition, we multiply equation (3.9) by the weight $(1-x)$ and we integrate the result over $[0,1]$ with respect to x to obtain

$$\int_0^1 (1-x)u(x,t)dx = \frac{k}{v_0}\int_0^1 (1-x)v(x,t)dx - k \qquad \dots (26)$$

which is the weight average entropy. Then, instead of searching for a pair of function (v,z), a solution of the problem given by equation (11)-(17), is made by searching for the function u, solution of problem given by equation (20)-(23), then the solution will be $v = u + z$.

4. The Variational Iteration Method for Solving A Nonlocal Problem Arising in Thermoelasticity [4]

The variational iteration method will be used to solve the nonlocal problem arising in thermoelasticity.

To do this considers the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \le x \le 1, \qquad 0 \le t \le 1 \qquad \dots (27)$$

Together with initial condition

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1 \qquad \dots (28)$$

And the homogeneous nonlocal conditions:

$$\int_0^1 u(x,t)dx = \theta_1(t), \qquad 0 \le t \le T \qquad \dots (29)$$

and

$$\int_0^1 xu(x,t)dx = \theta_2(t), \qquad 0 \le t \le T \qquad \dots (30)$$

As mentioned above, this nonlocal problem is transformed to an equivalent one with homogeneous nonlocal conditions by using the transformation:

$$v(x,t) = u(x,t) - z(x,t), \qquad 0 \le x \le 1, \qquad 0 \le t \le T$$

where z is defined previously.

Then the function v is seen to be the solution of the partial differential equation:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + g(x,t), \qquad 0 \le x \le 1, \qquad 0 \le t \le T \qquad \dots (31)$$

together with initial condition

$$v(x,0) = m(x), \qquad 0 \le x \le 1 \qquad \dots (32)$$

and the homogeneous nonlocal conditions:

$$\int_0^1 v(x,t)dx = 0, \qquad 0 \le t \le T \qquad \ldots (33)$$

and

$$\int_0^1 xv(x,t)dx = 0, \qquad 0 \le t \le T \qquad \ldots (34)$$

where

$$g(x,t) = f(x,t) - \frac{\partial z(x,t)}{\partial t}$$

$$m(x) = u_0(x) - z(x,0)$$

According to the variational iteration method, we consider the correction functional in t direction for equation (31) in the following form:

$$v_{i+1}(x,t) = v_i(x,t) + \int_0^t \lambda(s)\left[\frac{\partial v_i(x,s)}{\partial s} - \frac{\partial^2 \tilde{v}_i(x,s)}{\partial x^2} - g(x,s)\right]ds \qquad i = 0,1,\ldots \qquad \ldots (35)$$

where λ is the generalized Lagrange multiplier. Thus by taking the variation of above equation one can have:

$$\delta v_{i+1}(x,t) = \delta v_i(x,t) + \delta\int_0^t \lambda(s)\left[\frac{\partial v_i(x,s)}{\partial s} - \frac{\partial^2 v_i(x,s)}{\partial x^2} - g(x,s)\right]ds$$

Thus by using the integration by part, the above equation becomes:

$$\delta v_{i+1}(x,t) = \delta v_i(x,t) + \delta\lambda(s) v_i(x,s)\Big|_{s=t} - \delta\int_0^t \left[\lambda'(s) v_i(x,s) + \lambda(s)\frac{\partial^2 \tilde{v}_i(x,s)}{\partial x^2} + \lambda g(x,s)\right]ds$$

$$= \left[1 + \lambda(s)\big|_{s=t}\right]\delta v_i(x,s) - \delta\int_0^t \left[\lambda'(s) v_i(x,s) + \lambda(s)\frac{\partial^2 \tilde{v}_i(x,s)}{\partial x^2} + \lambda g(x,s)\right]ds$$

The stationary condition would be as follows:

$$1 + \lambda(s)\big|_{s=t} = 0, \qquad 0 \le s \le t$$

and

$$\lambda'(s) = 0.$$

Thus

$$\lambda(s) = -1.$$

Therefore the iterative formula for computing $v_i(x,t)$ taking the form:

$$v_{i+1}(x,t) = v_i(x,t) - \int_0^t \lambda(s)\left[\frac{\partial v_i(x,s)}{\partial s} - \frac{\partial^2 v_i(x,s)}{\partial x^2} - g(x,s)\right]ds \qquad i = 0,1,\cdots \qquad \ldots (36)$$

For simplicity, let $v_0(x,t) = m(x)$, then

$$\int_0^1 v_0(x,t)dx = \int_0^1 m(x)dx = \int_0^1 [u_0(x) - z(x,0)]dx = \int_0^1 u_0(x)dx - \int_0^1 z(x,0)dx$$

$$= \theta_1(0) - \int_0^1 [12\theta_2(0) - 6\theta_1(0)]xdx + \int_0^1 [6\theta_2(0) - 4\theta_1(0)]dx$$

$$= \theta_1(0) - 6\theta_2(0) + 3\theta_1(0) + 6\theta_2(0) - 4\theta_1(0)$$

$$= 0$$

and,

$$\int_0^1 xv_0(x,t)dx = \int_0^1 xm(x)dx = \int_0^1 x[u_0(x) - z(x,0)]dx = \int_0^1 xu_0(x)dx - \int_0^1 xz(x,0)dx$$

$$= \theta_1(0) - \int_0^1 [12\theta_2(0) - 6\theta_1(0)]x^2dx + \int_0^1 [6\theta_2(0) - 4\theta_1(0)]xdx$$

$$= \theta_2(0) - 4\theta_2(0) + 2\theta_1(0) + 3\theta_2(0) - 2\theta_1(0)$$

$$= 0$$

Then, any initial condition $v_0(x,t)$ given by equation (32) must satisfy the homogeneous nonlocal conditions (33)-(34) help to starting with. Then by substituting $i = 0$ into equation (36) one can get:

$$v_1(x,t) = v_0(x,t) - \int_0^t \left[\frac{\partial v_0(x,s)}{\partial s} - \frac{\partial^2 v_0(x,s)}{\partial x^2} - g(x,s)\right]ds$$

$$= m(x) - \int_0^t \left[-\frac{\partial^2 m(x)}{\partial x^2} - g(x,s)\right]ds$$

$$= m(x) + tm''(x) + \int_0^t \left[f(x,s) - \frac{\partial z(x,s)}{\partial s}\right]ds$$

$$= m(x) + tm''(x) + \int_0^t f(x,s)ds - z(x,t) + z(x,0)$$

$$= u_0(x) + tm''(x) + \int_0^t f(x,s)ds - z(x,t)$$

To illustrate this method, consider the following example:

Example

Consider the following one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2tx - 2, \quad 0 \le x \le 1, \quad 0 \le t \le 1 \qquad \dots (37)$$

Together with the initial condition:

$$u(x,0) = x^2 \qquad \dots (38)$$

and the nonhomogeneous nonlocal conditions

$$\int_0^1 u(x,t)dx = \frac{1}{3} + \frac{t^2}{2} \qquad \dots (39)$$

and

$$\int_0^1 xu(x,t)dx = \frac{1}{4} + \frac{t^2}{3} \qquad \dots (40)$$

It is clear that

$$\theta_1(0) = \frac{1}{3} = \int_0^1 u_0(x)ds = \int_0^1 x^2 dx$$

and

$$\theta_2(0) = \frac{1}{4} = \int_0^1 xu_0(x)dx = \int_0^1 x^3 dx$$

That is the compatibility conditions are satisfied. To solve such problem by using the variational iteration method, we must transform it into an equivalent problem given by equations (31)-(34) with homogeneous nonlocal conditions.

In this case:

$$z(x,t) = (1+t^2)x - \frac{1}{6}$$

$$g(x,t) = -2$$

and

$$m(x) = x^2 - x + \frac{1}{6}.$$

Therefore the nonlocal problem given by equation (37)-(40) becomes:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - 2, \qquad 0 \le x \le 1, \qquad 0 \le t \le 1 \qquad \dots (41)$$

together with the initial condition:

$$v(x,0) = x^2 - x + \frac{1}{6}, \qquad 0 \le x \le 1 \qquad \dots (42)$$

and the homogeneous nonlocal conditions

$$\int_0^1 v(x,t)dx = \int_0^1 xv(x,t)dx = 0, \qquad 0 \le t \le 1 \qquad \dots (43)$$

Let

$$v_0(x,t) = m(x) = x^2 - x + \frac{1}{6}$$

then

$$\frac{\partial v_0}{\partial s} = 0$$

and

$$\frac{\partial^2 v_0}{\partial x^2} = 2.$$

Hence

$$v_1(x,t) = v_0(x,t) - \int_0^t \left[\frac{\partial v_0(x,s)}{\partial s} - \frac{\partial^2 v_0(x,s)}{\partial x^2} - g(x,s) \right] ds$$

$$= x^2 - x + \frac{1}{6} - \int [0 - 2 + 2] ds$$

$$= x^2 - x + \frac{1}{6}$$

Therefore

$$v_i(x,t) = v_0(x,t) = x^2 - x + \frac{1}{6} \qquad i = 1,2,\ldots$$

and this implies that

$$v(x,t) = \lim_{i\to\infty} v_i(x,t) = x^2 - x + \frac{1}{6}$$

Therefore the solution of the original problem is

$$u(x,t) = v(x,t) + z(x,t)$$

$$= x^2 - x + \frac{1}{6} + (1+t^2)x - \frac{1}{6}$$

$$= x^2 - x + \frac{1}{6} + x + t^2 x - \frac{1}{6}$$

$$= x^2 + t^2 x$$

which is the exact solution for the original nonlocal problem.

Example

Consider the following one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + (1-t)e^{-x}, \qquad 0 \le x \le 1, \qquad 0 \le t \le 1$$

together with the initial condition:

$$u(x,0) = 0, \qquad 0 \le x \le 1$$

and the nonhomogeneous nonlocal conditions:

$$\int_0^1 u(x,t)dx = (1-e^{-1})t, \qquad 0 \le t \le 1$$

and

$$\int_0^1 xu(x,t)dx = (1-2e^{-1})t, \qquad 0 \le t \le 1$$

That is the compatibility conditions are satisfied. To solve such problem by using the variational iteration method, we must transform it into an equivalent problem given by equations (31)-(34) with homogeneous nonlocal conditions.

It is clear that

$$\theta_1(0) = \int_0^1 u_0(x)ds = 0$$

and

$$\theta_2(0) = \int_0^1 xu_0(x)dx = 0$$

In this case:

$$z(x,t) = 6(1-3e^{-1})tx - 2(1-4e^{-1})t$$

$$z(x,0) = 0$$

$$\frac{\partial z}{\partial t} = 6(1-3e^{-1})x - 2(1-4e^{-1})$$

$$g(x,t) = e^{-x}(1-t) - 6(1-3e^{-1})x + 2(1-4e^{-1})$$

$$m(x) = 0 - z(x,0) = 0$$

$$v_1(x,t) = \int_0^t \left[e^{-x}(1-s) - 6(1-3e^{-1})x + 2(1-4e^{-1})\right]ds$$

$$= e^{-x}(t - \frac{t^2}{2}) - 6(1-3e^{-1})xt + 2(1-4e^{-1})t$$

$$v_2(x,t) = e^{-x}(t - \frac{t^2}{2}) - 6(1-3e^{-1})xt + 2(1-4e^{-1})t -$$

$$\int_0^t \left[e^{-x}(1-s) - 6(1-3e^{-1})x + 2(1-4e^{-1}) - e^{-x}(s - \frac{s^2}{2}) + 6(1-3e^{-1})x - 2(1-4e^{-1}) \right] ds$$

$$v_2(x,t) = e^{-x}(t - \frac{t^3}{6}) - 6(1-3e^{-1})xt + 2(1-4e^{-1})t$$

By continuing in this manner one can get

$$v_i(x,t) = e^{-x}\left[t - \frac{t^{i+1}}{(i+1)!} \right] - 6(1-3e^{-1})xt + 2(1-4e^{-1})t, \qquad i = 1,2,\ldots$$

Hence,

$$v(x,t) = \lim_{i\to\infty} v_i(x,t) = te^{-x} - 6(1-3e^{-1})xt + 2(1-4e^{-1})t$$

In this case

$$u(x,t) = v(x,t) + z(x,t)$$
$$= te^{-x} - 6(1-3e^{-1})xt + 2(1-4e^{-1})t + 6(1-3e^{-1})xt - 2(1-4e^{-1})t$$
$$= te^{-x}$$

Which is the exact solution of the original problem.

Conclusions

In the application of the Variational Iterational Method, it was noted that every initial approximation to the solution of the non-local problems must satisfy the local and non-local conditions that associated with these problems. So, one can easily use the Variational Iterational Method to solve the one dimensional non-homogeneous Laplace equation with nonhomogeneous nonlocal conditions.

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